Chapter 3 – Polygons and Quadrilaterals

The Star of Lakshmi is an eight pointed star in Indian philosophy that represents the eight forms of the Hindu goddess Lakshmi. Lakshmi is the goddess of good fortune and prosperity. Of importance to us however, is not the philosophical meaning of the star (although it is very interesting!) but rather that the symbol is an 8 pointed star – otherwise known as a complex star, formed by the intersection of two squares. In the picture to the left we see that this joining of squares forms both regular and irregular shapes. These shapes are called polygons and quadrilaterals. In this chapter we explore the properties of polygons and then narrow our focus to quadrilaterals.

Concepts & Skills

The skills and concepts covered in this chapter include:

- Define, identify, and classify polygons and quadrilaterals
- Differentiate between convex and concave shapes
- Find the sums of interior angles in convex polygons.
- Identify the special properties of interior angles in convex quadrilaterals.
- Define a parallelogram.
- Understand the properties of a parallelogram
- Apply theorems about a parallelogram’s sides, angles and diagonals.
- Prove a quadrilateral is a parallelogram in the coordinate plane.
- Define and analyze a rectangle, rhombus, and square.
- Determine if a parallelogram is a rectangle, rhombus, or square in the coordinate plane.
- Analyze the properties of the diagonals of a rectangle, rhombus, and square.
- Define and find the properties of trapezoids, isosceles trapezoids, and kites.
- Discover the properties of mid-segments of trapezoids.
- Plot trapezoids, isosceles trapezoids, and kites in the coordinate plane.
Section 3.1 - Classifying Polygons

INSTANT RECALL!

1. Where have you seen 4, 5, 6 or 8-sided polygons in real life? List 3 examples.
2. Fill in the blank.
   a. Vertical angles are always ________________.
   b. Linear pairs are ____________________.
   c. The parts of an angle are called _______________ and a _______________.

3.1a Polygons

A Polygon is defined as any closed planar figure that is made entirely of line segments that intersect at their endpoints.

**Characteristics of Polygons Include:**

- They can have any number of sides and angles, but the sides can never be curved.
- The sides of the polygons are called segments and the points where the segments intersect are called vertices.

**IMPORTANT TO NOTE:** The easiest way to identify a polygon is to look for a closed figure with no curved sides.

Based on the definition above, which of the figures below fits the definition of a polygon?

A  B  C  D
Which of the figures below is not a polygon?

A regular polygon is a polygon that has all equal sides and all equal angles. The standard shapes such as a square or equilateral triangle are examples of regular polygons.
3.1b Convex and Concave Polygons

Polygons are further defined as being convex or concave. Think of the term concave as referring to a cave, or "caving in". A concave polygon has a section that "points inward" toward the middle of the shape. All stars are concave polygons.

**IMPORTANT TO NOTE:** One or more of a concave polygon’s interior angles will be greater than 180°.

Each of the following is an example of a concave polygon.
Convex polygons are polygons that have all of its vertices pointing “out”. These polygons tend to be our more common shapes – squares, triangles, rectangles, pentagons, etc.

**IMPORTANT TO NOTE:** All of the interior angles of a convex polygon will always be less than 180˚.

Each of the following is an example of a convex polygon.

![Convex Polygons](image)

### 3.1c Diagonals

Diagonals are line segments that link two non-adjacent vertices in a given polygon. As we see in the diagram below, the red lines are all considered diagonals of the pentagon. In this example, the pentagon has 5 diagonals. All pentagons will have five diagonals.
Diagonals: Convex vs. Concave Polygons

a) Diagonals in convex polygons, such as the pentagon above, will always intersect the polygon at two points (vertices). They will also always lie on the inside of a convex polygon.

b) Diagonals in concave polygons can lie both inside and outside of the polygon. They may also intersect the polygon at more than two points. We can see in the diagram below the convex polygon has 4 internal diagonals but also one diagonal that lies outside of the polygon.

![Diagram of a convex polygon with internal and one external diagonal]

**IMPORTANT TO REMEMBER:** Diagonals connect non-adjacent vertices!

Finding the Number of Diagonals in a Polygon

Regardless of the polygon being concave or convex, there are two methods for finding the number of diagonals present in a polygon. First, you can simply draw the diagonals based on your understanding of the definition. While this may seem easy to do, it does allow room for human error as one vertex may have a diagonal that connects to more than one non-adjacent vertex. Think of our pentagon, each vertex has a diagonal that connect to two non-adjacent vertices. It could be easy to leave one or more of the diagonals out!

![Diagram of a pentagon with all diagonals]

As one may suspect, there is a mathematic way to determine the number of diagonals a polygon has. Again, this applies to both convex and concave
polygons. The formula to determine the number of distinct diagonals a polygon has is

$$\text{Number of Distinct Diagonals} = \frac{n(n-3)}{2}$$

Where \( n \) is the number of vertices the polygon has

**Example:** How many diagonals does a Heptagon (7-sided) polygon have?

As you can see, in order to use the first method, you would first need to draw a Heptagon and then draw its diagonals. This can get messy!

Using our diagonal formula however we can calculate the number of diagonals

$$\text{Number of Diagonals in a Heptagon} = \frac{n(n-3)}{2} = \frac{7(7-3)}{2} = 14$$

**YOU TRY!**

Calculate the number of diagonals for each of the following polygons. You may try both methods if you would like but verify your answer with the formula!

1. Triangle
2. Square
3. Pentagon
4. Hexagon (6sides)
5. Octagon (8sides)
3.1d - Naming Polygons

Polygon’s can be broken into three distinct classes. Each of these classes can be broken down further with shapes being identified by the number of sides they have. This is shown in the chart below.

**IMPORTANT TO NOTE:** Whether a polygon is convex or concave, it can always be named by its number of sides. See the chart below.

<table>
<thead>
<tr>
<th>Polygon Name</th>
<th>Number of Sides</th>
<th>Number of Diagonals</th>
<th>Convex Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>0</td>
<td><img src="triangle.png" alt="Triangle" /></td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>2</td>
<td><img src="quadrilateral.png" alt="Quadrilateral" /></td>
</tr>
<tr>
<td>Polygon Name</td>
<td>Number of Sides</td>
<td>Number of Diagonals</td>
<td>Convex Example</td>
</tr>
<tr>
<td>----------------------------------</td>
<td>-----------------</td>
<td>---------------------</td>
<td>----------------</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Heptagon</td>
<td>7</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>Octagon</td>
<td>8</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Nonagon</td>
<td>9</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Decagon</td>
<td>10</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Undecagon or hendecagon</td>
<td>11</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Dodecagon</td>
<td>12</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Polygon Name</td>
<td>Number of Sides</td>
<td>Number of Diagonals</td>
<td>Convex Example</td>
</tr>
<tr>
<td>--------------</td>
<td>-----------------</td>
<td>---------------------</td>
<td>----------------</td>
</tr>
<tr>
<td>$n$-gon</td>
<td>$n$ (where $n &gt; 12$)</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

**Internet & Video Links**

For more information on the above topics visit the following:

1. [http://www.mathsisfun.com/shape.html](http://www.mathsisfun.com/shape.html)
2. [http://www.mathsisfun.com/geometry/polygons.html](http://www.mathsisfun.com/geometry/polygons.html)
Section 3.1 Homework Questions

In problems 1-6, name each polygon in as much detail as possible and calculate its number of diagonals using the diagonal formula.
6. Explain why the following figures are NOT polygons:

7. How many diagonals can you draw from **one vertex** of a pentagon? Draw a sketch of your answer.

8. How many diagonals can you draw from **one vertex** of an octagon? Draw a sketch of your answer.

9. How many diagonals can you draw from **one vertex** of a dodecagon?
Section 3.2: Angle Properties of Polygons

3.2a Interior and Exterior Angles

Aside from having sides, vertices, and diagonals, all polygons also have interior and exterior angles. As we see in the diagram below, for all convex polygons, the sum of an interior and exterior angle is 180° making them supplementary angles. If you recall from integrated math 1, since these angles share a common side we can also classify them as a linear pair.

$120° + 60° = 180°$

Convex Polygon with Interior/Exterior Angles

Concave polygons also have interior and exterior angles that add up to 180°, but for the angle that is concave, you need to remember that once angle will have a negative value and one will be greater than 180°. We can see this in the diagram below.

$230° + (-50°) = 180°$

Concave Polygon with Interior/Exterior Angles
3.2b Interior Angles of a Polygon

As stated in the previous section, regular polygons are made of all equal angles and equal sides. The interior angles are those angles that are on the inside of the polygon. No matter the type of polygon you are working with, the sum of all of the interior angles will always add up to one constant value. This value depends on the number of sides a polygon has. For example, any four sided polygon will always have interior angles that add up to 360° no matter if it is a regular (square) polygon, or an irregular polygon. As well, concavity and convexity do not impact this fact!

\[
4 \times 90 = 360
\]

92 + 108 + 77 + 83 = 360

Regular Polygon

Irregular Polygon

To find the sum of the interior angles for any polygon you need to use the following formula

\[
\text{Sum of Interior Angles} = 180^\circ \times (n-2); \text{ where } n = \text{ the number of sides of the polygon.}
\]
Investigation 3-1: Sum of Interior Angles

Tools Needed: paper, pencil, ruler, colored pencils (optional)

1. Draw a quadrilateral, pentagon, and hexagon.

2. Cut each polygon into triangles by drawing all the diagonals from one vertex. Count the number of triangles.

Make sure none of the triangles overlap.

3. Make a table with the information below.

<table>
<thead>
<tr>
<th>Name of Polygon</th>
<th>Number of Sides</th>
<th>Number of $\Delta$s from one vertex</th>
<th>$(Column 3) \times (\text{in a } \Delta)$</th>
<th>Total Number of Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>2</td>
<td>$2 \times 180^\circ$</td>
<td>360°</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>3</td>
<td>$3 \times 180^\circ$</td>
<td>540°</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>4</td>
<td>$4 \times 180^\circ$</td>
<td>720°</td>
</tr>
</tbody>
</table>

4. Do you see a pattern? Notice that the total number of degrees goes up by $180^\circ$. So, if the number sides is $n$, then the number of triangles from one vertex is $(n-2)$. Therefore, the formula would be $(n-2) \times 180^\circ$. 
You Try!

1. Find the sum of the interior angles of an octagon.
2. The sum of the interior angles of a polygon is $1980^\circ$. How many sides does this polygon have?
3. How many degrees does each angle in an equiangular nonagon have?
4. Algebra Connection Find the value of $x$ and the measure of each angle in the diagram below.

3.2c Exterior Angles of a Polygon

Regardless of the number of sides of the polygon or whether it is regular or irregular, concave or convex, the sum of any polygon’s exterior angles will ALWAYS be $360^\circ$!

$$71^\circ + 34^\circ + 58^\circ + 82^\circ + 34^\circ + 81^\circ = 360^\circ$$
In the diagram above we see both a regular and irregular hexagon. We also see that while the exterior angles of the irregular hexagon may be different values, just like the regular hexagon, they all add up to 360°.

To determine the measure of the exterior angles for a regular polygon, all you need to do is to divide 360° by the number of sides of the polygon.

For example, for a dodecagon (12-sided polygon) \(360/12 = 30°\). So, all of the exterior angles of a regular dodecagon will measure 30°.

### Investigation 3-2: Exterior Angle Tear-Up

Tools Needed: pencil, paper, colored pencils, scissors

1. Draw a hexagon like the hexagons above. Color in the exterior angles as well.
2. Cut out each exterior angle and label them 1-6.
3. Fit the six angles together by putting their vertices together. What happens?

The angles all fit around a point, meaning that the exterior angles of a hexagon add up to 360°, just like a triangle. We can say this is true for all polygons.

- Exterior Angle Sum Theorem: The sum of the exterior angles of any polygon is 360°.

**You try!**

Find the measure of the exterior angles for the following regular polygons.

- Triangle, pentagon, hexagon, heptagon, octagon, decagon
3.2d Algebraically finding Interior and Exterior Angles

Recall from Integrated Math 1 (Ch. 6) we looked at how to find the measures of angles that formed vertical, complementary, and supplementary relations.

**INSTANT RECALL!**

Find the value of $x$.

1) 

![Diagram 1: Two intersecting lines with angles $(5x + 4)^\circ$ and $39^\circ$.]

2) 

![Diagram 2: A triangle with angles $3x^\circ$, $63^\circ$, and another angle.]

Find the measure of angle $b$.

3) 

![Diagram 3: Two intersecting lines with angles $60^\circ$ and $b$.]

4) 

![Diagram 4: Two intersecting lines with an angle $123^\circ$.]
We can apply the same logical to finding the measures of interior and exterior angles of a polygon. Let’s consider the following diagram.

We first identify that the diagram is that of a triangle (polygon) with an extended side ray. This ray helps remind us that the interior and exterior angles formed at this point in the diagram are supplementary (add up to 180°) thus proving the relationship we established in the previous section about the measure of interior and exterior angles in a polygon. Furthermore, we also know that the sum of the interior angles of a triangle has to equal 180°. We can calculate this using our interior angle formula 180(n-2).

For any triangle the sum of the interior angles = 180°(3-2)= 180°.

You need to decide at this point about the amount of information you have and whether you can work on the interior or exterior of the polygon. Clearly, there is not enough information to work on the exterior angles so begin with the interior angles.

We know 72° + (7x+3) + (3x+5) = 180°

Solve for x and we get that x = 10 therefore, aside from the 72° angle in the triangle, there are angles of (7(10) + 3) = 73° and then (3(10)+5) = 35°.

To find the measure of the exterior angle in this diagram, we know that one of the angles of the linear pair has a measure of 35° (as solved for above) and that 35° + (9y+1)°=180°. Solving for y we see that the measure of the exterior angle is 145°.
You Try!

1. Find the values of x and y and then find the measure of each of the angles in the diagram below. Identify the interior and exterior angles by their measure.

2. 
   a. Find \( w^\circ, x^\circ, y^\circ, \) and \( z^\circ \).
   b. What is \( w^\circ + y^\circ + z^\circ \)?
   c. What two angles add up to \( y^\circ \)?
   d. What are \( 72^\circ, 59^\circ \), and \( x^\circ \) called? What are \( w^\circ, y^\circ, \) and \( z^\circ \) called?

3. What is \( y^\circ \)?

4. What is the measure of each exterior angle of a regular heptagon?
Internet & Video Links

For more information on the above topics visit the following:

### Section 3.2 Homework Questions

1. Fill in the table.

<table>
<thead>
<tr>
<th># of sides</th>
<th># of $\triangle$s from one vertex</th>
<th>$\triangle s \times 180^\circ$ (sum)</th>
<th>Each angle in a regular $n$-gon</th>
<th>Sum of the exterior angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>180°</td>
<td>60°</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>360°</td>
<td>90°</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>540°</td>
<td>108°</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>720°</td>
<td>120°</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
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<td></td>
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<td>9</td>
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<td>10</td>
<td></td>
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</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. What is the sum of the angles in a 15-gon?
3. What is the sum of the angles in a 23-gon?
4. The sum of the interior angles of a polygon is $4320^\circ$. How many sides does the polygon have?
5. The sum of the interior angles of a polygon is $3240^\circ$. How many sides does the polygon have?
6. If the sum of the interior angles is 20 right angles, find the number of sides the polygon has.
7. What is the measure of each angle in a regular 16-gon?
8. What is the measure of each angle in an equiangular 24-gon?
9. What is the measure of each exterior angle of a dodecagon?
10. What is the measure of each exterior angle of a 36-gon?
11. What is the sum of the exterior angles of a 27-gon?
12. If the measure of one interior angle of a regular polygon is 160°, how many sides does it have?
13. How many sides does a regular polygon have if the measure of one of its interior angles is 168°?
14. How many sides does a regular polygon have if the measure of one exterior angle is 15°?
15. If the measure of one exterior angle of a regular polygon is 36°, how many sides does it have?

For questions 16-25, find the measure of the missing variable(s).

16. 

17. 

18. 

19.
26. The interior angles of a pentagon are $x^\circ, x^\circ, 2x^\circ, 2x^\circ$, and $2x^\circ$. What is the measure of the larger angles?

27. The exterior angles of a quadrilateral are $x^\circ, 2x^\circ, 3x^\circ$, and $4x^\circ$. What is the measure of the smallest angle?

28. The interior angles of a hexagon are $x^\circ, (x + 1)^\circ, (x + 2)^\circ, (x + 3)^\circ, (x + 4)^\circ$, and $(x + 5)^\circ$. What is $x$?

29. Challenge Each interior angle forms a linear pair with an exterior angle. In a regular polygon you can use two different formulas to find the measure of each exterior angle. One way is $\frac{360^\circ}{n}$ and the other is $180^\circ - \frac{(n-2)180^\circ}{n}$ (180$^\circ$ minus Equiangular Polygon Formula). Use algebra to show these two expressions are equivalent.

30. Angle Puzzle Find the measures of the lettered angles below given that $m \parallel n$. 
Section 3.3: Quadrilaterals

3.3a Properties of Quadrilaterals

INSTANT RECALL!

1. What is a bisector?
2. What is a perpendicular bisector? How do we know two lines are perpendicular?

A quadrilateral is any four-sided polygon. The primary quadrilaterals can be seen in the chart below.

- Parallelogram
- Square
- Rectangle
- Trapezoid
- Rhombus
- Kite

A Parallelogram is a quadrilateral in which

a) opposite sides are parallel
b) opposite sides are equal
c) opposite angles are equal
d) diagonals bisect each other

To continue to explore the properties of a parallelogram, see the website:

A **Rectangle** is a quadrilateral having all of the properties of a parallelogram but also

a) four right angles but unequal sides  
b) diagonals that are equal

A **Square** is a special rectangle in which all the sides are equal. All interior angles are congruent (90°).

**IMPORTANT TO NOTE:** A square can be classified as a rectangle, square, and a rhombus as it holds all of the properties of these quadrilaterals.

A **Trapezoid** is a quadrilateral with

a) one pair of parallel sides  
b) Two pair of adjacent side congruent angles  
c) Diagonals that have congruent pairs

A **Rhombus** is a quadrilateral that has all of the properties of a parallelogram but also

a) all sides are equal  
b) it has two congruent interior angle pairs  
c) its diagonals are bisectors of opposite angles of the figure

**IMPORTANT TO NOTE:** A rhombus is also a kite
A **Kite** is a quadrilateral with

a) two pairs of equal adjacent sides  
b) diagonals are perpendicular to one-another  
c) there is a pair of equal opposite angles  
d) there is one diagonal which bisects a pair of opposite angles

**Internet & Video Links**

3. [http://www.mathsisfun.com/geometry/parallelogram.html](http://www.mathsisfun.com/geometry/parallelogram.html)  
4. [http://www.mathsisfun.com/geometry/rectangle.html](http://www.mathsisfun.com/geometry/rectangle.html)  
5. [http://www.mathsisfun.com/geometry/square.html](http://www.mathsisfun.com/geometry/square.html)  
7. [http://www.mathsisfun.com/geometry/rhombus.html](http://www.mathsisfun.com/geometry/rhombus.html)  
11. [http://www.mathsisfun.com/geometry/kite.html](http://www.mathsisfun.com/geometry/kite.html)  
3.3b Applying the Properties of Parallelograms

Recall from the previous section the following properties of a parallelogram:

A Parallelogram is a quadrilateral in which

   a) opposite sides are parallel
   b) opposite sides are equal
   c) opposite angles are equal
   d) diagonals bisect each other

In this section we will use these properties to solve problems related to parallelograms. It is important to know the properties in order to solve each of the problems we will encounter.

Find the Interior Angles of a Parallelogram. ABCD is a parallelogram. If \( \angle D = 140^\circ \), find the measure of the other three interior angles.

Using the properties of parallelograms we know that they have opposite angles that are equal therefore if \( \angle D = 140^\circ \) then \( \angle B = 140^\circ \). This leaves us with having to find \( \angle A \) and \( \angle C \) - which will also be equal since they are opposite one another. To find these measures recall from section 3.1

\[
\text{Sum of Interior Angles} = 180^\circ \times (n-2); \text{ where } n = \text{ the number of sides of the polygon.}
\]

Given this, a parallelogram (as will all quadrilaterals) will have a sum of interior angles of \( 360^\circ \). As a result we can establish that

\[
m\angle A + m\angle B + m\angle C + m\angle D = 360^\circ
\]

Using this as a guide we can write

\[
140^\circ + 140^\circ + 2x = 360^\circ \text{ (we use } 2x \text{ because the two angles are equal)}
\]

Solving this equation we get an the \( m\angle A = 40^\circ \) and \( m\angle C = 40^\circ \)
**Find the Length of a Side of a Parallelogram:** VWXY is a parallelogram. If line segment XW = 19, find the value of x for line segment YV to satisfy the properties of a parallelogram.

![Parallelogram diagram](image)

Recall that in order for a shape to be a parallelogram, it must have two parallel sides of equal lengths. In the diagram above XW is parallel to YV therefore the length of YV must equal that of XW. To solve this we simply set up the equation XW = YV and then substitute the values that we know. Doing this gives us

\[
XW = YV \\
19 = 2x + 7 \\
X = 6
\]

**Find the Length of a Parallelograms Diagonal:** PQRS is a parallelogram.

![Parallelogram diagram](image)

Recall that one of the properties of the parallelogram is that the diagonals bisect one-another. This means that each diagonal “cuts through” the other diagonal at its midpoint creating two congruent halves. In the diagram above we know because of this property QM = MS and that PM = RM. As a result, the find the value for x, as we did in the previous example, we just have to set the two segments equal to one-another and solve for the missing value.

\[
QM = MS \quad \text{therefore} \quad 10 = 2x - 10 \quad \text{where} \quad X = 10
\]
You Try!

1. Use the properties of parallelograms to find the values of $x$ and $y$.

   \[ \begin{align*}
   \text{L} & : 6x - 7 \\
   \text{M} & : y^2 + 3 \\
   \text{O} & : \text{12} \\
   \text{N} & : 2x + 9
   \end{align*} \]

2. SAND is a parallelogram where $SY = 4x - 11$ and $YN = x + 10$. Solve for $x$.

   \[ \begin{align*}
   \text{S} & : \text{Y} \\
   \text{A} & \\
   \text{D} & \\
   \text{N} & \\
   \end{align*} \]

3. Use the diagram below

   Find $m\angle GED$

   \[ \begin{align*}
   \text{E} & : 4x - 5 \\
   \text{D} & : 53^\circ \\
   \text{F} & : 8x \\
   \text{G} & : \text{8}x
   \end{align*} \]

4. Use the diagram below

   Find $m\angle T$

   \[ \begin{align*}
   \text{S} & : 9x \\
   \text{R} & : 51^\circ \\
   \text{I} & : 13x - 3
   \end{align*} \]
Section 3.3b Homework Questions

1. If $m\angle B = 72\degree$ in parallelogram $ABCD$, find the other three angles.
2. If $m\angle S = 143\degree$ in parallelogram $PQRS$, find the other three angles.
3. If $AB \perp BC$ in parallelogram $ABCD$, find the measure of all four angles.
4. If $m\angle F = x\degree$ in parallelogram $EFGH$, find expressions for the other three angles in terms of $x$.

For questions 5-13, find the measures of the variable(s). All the figures below are parallelograms.

5. 

6. 

7. 

8. 

9.
10. Use the parallelogram to find:

\[(4m - 8)° \quad (5n - 3)° \]
\[(3m + 17)° \quad (4n + 12)°\]

11. \[\begin{array}{c}
30 \\
8q \\
7q + 3 \\
4p - 2
\end{array}\]

12. \[\begin{array}{c}
6 \\
4s + 5 \\
6s + 1 \\
5r + 1
\end{array}\]

13. \[\begin{array}{c}
5t + 1 \\
7t - 5 \\
2u - 1
\end{array}\]

Use the parallelogram \( WAVE \) to find:

14. \( m\angle WAE \)
15. \( m\angle ESV \)
16. \( m\angle WEA \)
17. \( m\angle AVW \)
In the parallelogram $SNOW$, $ST = 6$, $NW = 4$, $m\angle OSW = 36^\circ$, $m\angle SNW = 58^\circ$, and $m\angle NTS = 80^\circ$. (diagram is not drawn to scale)

18. $SO$
19. $NT$
20. $m\angle NWS$
21. $m\angle SOW$

Plot the points $E(-1, 3), F(3, 4), G(5, -1), H(1, -2)$ and use parallelogram $EFGH$ for problems 22-25.

22. Find the coordinates of the point at which the diagonals intersect. How did you do this?
23. Find the slopes of all four sides. What do you notice?
24. Use the distance formula to find the lengths of all four sides. What do you notice?
25. Make a conjecture about how you might determine whether a quadrilateral in the coordinate is a parallelogram.

Use the diagram below to find the indicated lengths or angle measures for problems 29-32. The two quadrilaterals that share a side are parallelograms.

29. $w$
30. $x$
31. $y$
32. $z$
3.3c Showing a Quadrilateral is a Parallelogram in the Coordinate Plane

Know What? Four friends, Geo, Trig, Algie, and Calc are marking out a baseball diamond. Geo is standing at home plate. Trig is 90 feet away at 3rd base, Algie is 127.3 feet away at 2nd base, and Calc is 90 feet away at 1st base. The angle at home plate is 90˚, from 1st to 3rd is 90˚. Find the length of the other diagonal and determine if the baseball diamond is a parallelogram. If it is, what kind of parallelogram is it?

To show that a quadrilateral is a parallelogram in the coordinate plane, you will need to use a combination of the slope formulas, the distance formula and the midpoint formula in order to prove the properties of parallelograms correct.

- You would need to find the slope of all four sides to see if the opposite sides are parallel.
- You would have to find the length (using the distance formula) of each side to see if the opposite sides are congruent.
- You would need to use the midpoint formula for each diagonal to see if the midpoint is the same for both.
- Finally, you would need to show that one pair of opposite sides has the same slope (slope formula) and the same length (distance formula).
Let’s apply these steps to an example. Is the quadrilateral ABCD a parallelogram?

1. Identify the coordinates of the vertices A, B, C, and D.
   A= (-1,5), B=(3,3), C=(6,-4) and D=(2,-2)

2. Calculate the slope of each line segment to determine if the lines are parallel. If they are not you have proven that the shape is not a parallelogram so the problem is done.

   - Slope of AD = -7/3   slope of BC = -7/3
   - Slope of DC = -1/2   slope of AB = -1/2

   In this case both pairs of parallel lines have the same slope so they meet the first condition for being a parallelogram.

3. Using the distance formula, find the length of each line segment.
   Recall, \( D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \)

   - Distance of AD = \( \sqrt{(2 - (-1))^2 + (-2 - 5)^2} = \sqrt{58} \)
   - Distance of BC = \( \sqrt{(6 - 3)^2 + (-4 - 3)^2} = \sqrt{58} \)
   - Distance of DC = \( \sqrt{(6 - 2)^2 + (-4 - (-2))^2} = 2\sqrt{5} \)
   - Distance of AB = \( \sqrt{(3 - (-1))^2 + (3 - 5)^2} = 2\sqrt{5} \)
So far, each of the tests have pointed us in the direction to conclude that ABCD is a parallelogram but we still need to do one more!

4. Using the midpoint formula, determine if each diagonal has the same coordinates for its midpoint. Remember, this proves whether the diagonals are bisected or not!

Recall, Midpoint = \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)

Midpoint of AC = \( \left( \frac{6 + (-1)}{2}, \frac{5 + (-4)}{2} \right) = \frac{5}{2}, \frac{1}{2} \)

Midpoint of DB = \( \left( \frac{3 + 2}{2}, \frac{3 + (-2)}{2} \right) = \frac{5}{2}, \frac{1}{2} \)

Again we see that the diagonals share the same midpoint which proves that the lines are bisected into equal parts.

5. Conclusion. Each of the tests showed us algebraically that each of the properties of a parallelogram was met so therefore we can conclude with certainty that ABCD is a parallelogram!

WARNING: NEVER rely on your eyes to determine whether or not a shape is a parallelogram! You must apply the appropriate math tests!
You try!

Determine if the quadrilateral RSTU is a parallelogram?

1. Show that the diagonals of \(FGHJ\) bisect each other.

Know What? Revisited First, we can use the Pythagorean Theorem to find the length of the second diagonal.

\[
90^2 + 90^2 = d^2 \\
8100 + 8100 = d^2 \\
16200 = d^2 \\
d = 127.3
\]

This means that the diagonals are equal. If the diagonals are equal, the other two sides of the diamond are also 90 feet. Therefore, the baseball diamond is a parallelogram, and more specifically, it is a square.
Section 3.3c Homework Questions

For questions 1-3, determine the value of x and y that would make the quadrilateral a parallelogram.

1. \[ 8x - 5 \]
   \[ 3x + 20 \]

2. \[ \begin{align*}
   10x^\circ & \quad 10y^\circ \\
   100^\circ & \quad (11x - 8)^\circ
   \end{align*} \]

3. \[ \begin{align*}
   11 & \quad 15 \\
   5y & \quad 3x - 1
   \end{align*} \]

For questions 4-6, draw the quadrilateral on a coordinate plane and then determine if ABCD is a parallelogram.

4. \( A(8, -1), B(6, 5), C(-7, 2), D(-5, -4) \)
5. \( A(-5, 8), B(-2, 9), C(3, 4), D(0, 3) \)
6. \( A(-2, 6), B(4, -4), C(13, -7), D(4, -10) \)

The points \( Q(-1, 1), U(7, 1), A(1, 7) \) and \( D(-1, 5) \) are the vertices of quadrilateral \( QUAD \). Plot the points on graph paper to complete problems 7-10.

7. Find the midpoints of sides \( \overline{QU}, \overline{UA}, \overline{AD} \) and \( \overline{DQ} \). Label them \( W, X, Y \) and \( Z \) respectively.
8. Connect the midpoints to form quadrilateral \( WXYZ \). What does this quadrilateral appear to be?
9. Use slopes to verify your answer to problem 8.
10. Use midpoints to verify your answer to problem 8.
Section 3.4 Applying Properties of Rectangles, Rhombuses and Squares

INSTANT RECALL!

1. Define rectangle in your own words. Is a rectangle a parallelogram?
2. Define square in your own words. Is a square a parallelogram? Is it a rectangle?
3. List five examples where you might see a square, rectangle, or rhombus in real life.

Recall from section 3.3:

A Rectangle is a quadrilateral having all of the properties of a parallelogram but also
   a) four right angles but unequal sides
   b) diagonals that are equal

A Square is a special rectangle in which all the sides are equal. All interior angles are congruent (90°).

IMPORTANT TO NOTE: A square can be classified as a rectangle, square, and a rhombus as it holds all of the properties of these quadrilaterals.

A Rhombus is a quadrilateral that has all of the properties of a parallelogram but also
   a) all sides are equal
   b) it has two congruent interior angle pairs
   c) its diagonals are bisectors of opposite angles of the figure
3.4a Diagonals in Squares, Rectangles, & Rhombi

Recall from previous lessons that the diagonals in a parallelogram bisect each other. Therefore, the diagonals of a rectangle, square and rhombus also bisect each other. The diagonals of these parallelograms also have additional properties. These are:

- A parallelogram is a **rectangle** if and only if the diagonals are congruent.

- A parallelogram is a **rhombus** if and only if the diagonals are perpendicular and they bisect each angle.

**IMPORTANT TO NOTE:** There are no theorems about the diagonals of a square. We know that a square is a rhombus and a rectangle. So, the diagonals of a square have the properties of a rhombus and a rectangle.

Given the diagram, determine if the parallelogram is a square.

Based on the diagram we can conclude ABCD is a square because

a) All sides are congruent  
b) All interior angles are congruent  
c) The diagonals bisect one-another and are congruent

**You Try!** Which types of parallelogram are shown below? Justify your response.

a) ![Parallelogram](image1.png)  
b) ![Parallelogram](image2.png)
**3.4b Squares, Rectangles, & Rhombi in the Coordinate Plane**

To show that a parallelogram is a square, rectangle, or rhombus in the coordinate plane, you will need to use a combination of the slope formulas, the distance formula and the midpoint formula in order to prove the properties of each correct. This is pretty much the same thing we did in the section on proving parallelograms in the coordinate plane!

**Here are the steps to determine if a parallelogram is a, rectangle, rhombus, or square.**

1. See if the diagonals bisect each other by using the midpoint formula.
   - **Yes:** Parallelogram, continue to #2. **No:** A quadrilateral, done.

2. Determine if the diagonals are equal by using the distance formula.
   - **Yes:** Rectangle, skip to #4. **No:** Could be a rhombus, continue to #3.

3. Determine if the sides are congruent by using the distance formula.
   - **Yes:** Rhombus, done. **No:** Parallelogram, done.

4. See if the diagonals are perpendicular by finding their slopes.
   - **Yes:** Square, done. **No:** Rectangle, done.

Use the flow chart above to determine what type of parallelogram TUNE is if $T(0, 10)$, $U(4, 2)$, $N(-2, -1)$, and $E(-6, 7)$. 

![Diagram of parallelogram TUNE on the coordinate plane.](image-url)
At this point, you should be comfortable by not seeing the work so we will just go through the values here.

1. Using the midpoint formula and the vertices given, TN and EU both have midpoints at the coordinate of (-1,4.5) therefore we know we have a parallelogram (go to step #2)
2. Using the distance formula and the vertices given, the length of TN and EU is $5\sqrt{5}$. As a result we know that this is a rectangle. The next test we need to do is check to see if it is a square.
3. Using the slope formula we see that the slope of TN is $\frac{11}{2}$ while the slope of EU is $-\frac{1}{2}$ since the slopes are not negative reciprocals of one-another, we can determine that the shape is a rectangle.

**You try!** Determine is the quadrilateral in the diagram below is a square, rectangle, or rhombus

![Diagram of a quadrilateral](image)

**Know What? Revisited** In order for the patio to be a rectangle, first the opposite sides must be congruent. So, two sides are 21ft and two are 28 ft. To ensure that the parallelogram is a rectangle **without** measuring the angles, the diagonals must be equal. You can find the length of the diagonals by using the Pythagorean Theorem.

![Parallelogram with diagonals](image)

\[d^2 = 21^2 + 28^2 = 441 + 784 = 1225\]
\[d = \sqrt{1225} = 35 \text{ ft}\]
Section 3.4b Homework Questions

1. If RACE is a rectangle, find the information using the diagram below.

   a. $RG$
   b. $AE$
   c. $AC$
   d. $EC$
   e. $m\angle RAC$

2. If DIAM is a rhombus, find the information using the diagram below.

   a. $MA$
   b. $MI$
   c. $DA$
   d. $m\angle DIA$
   e. $m\angle MOA$

3. Draw a square and label it CUBE. Mark the point of intersection of the diagonals $Y$. Find:

   a. $m\angle UCE$
   b. $m\angle EYB$
   c. $m\angle UBY$
   d. $m\angle UEB$
For questions 4-12, determine if the quadrilateral is a parallelogram, rectangle, rhombus, square or none. Explain your reasoning.

4. 

5. 

6. 

7. 

8. 

9.
For questions 13-18 determine if the following are ALWAYS, SOMETIME, or NEVER true. Explain your reasoning.

13. A rectangle is a rhombus.
14. A square is a parallelogram.
15. A parallelogram is regular.
16. A square is a rectangle.
17. A rhombus is equiangular.
18. A quadrilateral is a pentagon.

For questions 19-22, determine what type of quadrilateral ABCD is.

19. $A(-2, 4), B(-1, 2), C(-3, 1), D(-4, 3)$
20. $A(-2, 3), B(3, 4), C(2, -1), D(-3, -2)$
21. $A(1, -1), B(7, 1), C(8, -2), D(2, -4)$
22. $A(10, 4), B(8, -2), C(2, 2), D(4, 8)$
For problems 23-25, find the value of each variable in the figures.

23.

24.

25.
Section 3.5: Applying Properties of Trapezoids and Kites

3.5a Trapezoids

A **Trapezoid** is a quadrilateral with

a) one pair of parallel sides
b) Two pair of adjacent side congruent angles
c) Diagonals that have congruent pairs

**Isosceles Trapezoid**: A trapezoid where the non-parallel sides are congruent.

Think of it as an isosceles triangle with the top cut off. Isosceles trapezoids also have parts that are labeled much like an isosceles triangle. Both parallel sides are called bases.

In an Isosceles Trapezoid, *The two angles along the same base in an isosceles trapezoid will also be congruent*. This creates two pairs of congruent angles.
Consider trapezoid TRAP below. What is $m\angle A$?

![Trapezoid TRAP](image)

Remember, the sum of the interior angles is $(4-2) \times 180^\circ = 360^\circ$. Therefore, $m\angle R = 115^\circ$ which means the measure of angles $P$ and $A$ are $65^\circ$.

### 3.5b Diagonals of a Trapezoid

Recall that the diagonals of a rectangle are congruent AND they bisect each other. The diagonals of an isosceles trapezoid are also congruent, but they do NOT bisect each other.

Using the diagram below prove that the diagonals of the isosceles trapezoid are congruent but do not bisect one-another.

![Diagram](image)

To do this, you will need to use the distance formula. First find the length of $BD$ and $AC$. Doing this we find that both are a length of $2\sqrt{10}$. By simply drawing the diagonals we see that they do not bisect one-another therefore, we have an isosceles trapezoid.
3.5c Mid-segment of a Trapezoid

The Mid-segment (of a trapezoid): A line segment that connects the midpoints of the non-parallel sides.

**IMPORTANT TO NOTE:** There is only one mid-segment in a trapezoid. It will be parallel to the bases because it is located halfway between them. Similar to the mid-segment in a triangle, where it is half the length of the side it is parallel to, the mid-segment of a trapezoid also has a link to the bases.

**Investigation 3-5: Mid-segment Property**

Tools Needed: graph paper, pencil, ruler

1. Draw a trapezoid on your graph paper with vertices $A(-1,5), B(2,5), C(6,1)$ and $D(-3,1)$. Notice this is NOT an isosceles trapezoid.

2. Find the midpoint of the non-parallel sides either by using slopes or the midpoint formula. Label them $E$ and $F$. Connect the midpoints to create the mid-segment.

3. Find the lengths of $AB$, $EF$, and $CD$. Can you write a formula to find the mid-segment?

- **Mid-segment Theorem:** The length of the mid-segment of a trapezoid is the average of the lengths of the bases, or $EF = \frac{AB + CD}{2}$. 

YOU TRY!

For each of the figures below find x. All figures are trapezoids with the mid-segment.

a) 

\[
\begin{array}{c}
12 \\
26 \\
x
\end{array}
\]

b) 

\[
\begin{array}{c}
x \\
24 \\
35
\end{array}
\]

c) 

\[
\begin{array}{c}
5x - 15 \\
20 \\
2x - 8
\end{array}
\]

http://www.brightstorm.com/math/geometry/polygons/trapezoid-midsegment-properties/

3.5d Kites

The last quadrilateral we will study is a kite. Like you might think, it looks like a traditional kite that is flown in the air!

- Kite: A quadrilateral with two sets of adjacent congruent sides.

A few examples:

![Kite examples](image)

IMPORTANT TO NOTE: A kite is the only quadrilateral that can be concave, as with the case of the last kite. If a kite is concave, it is called a dart.
The angles between the congruent sides are called *vertex angles*. The other angles are called *non-vertex angles*. If we draw the diagonal through the vertex angles, we would have two congruent triangles.

**Other Characteristics of a Kite:**

- The non-vertex angles of a kite are congruent.
- The diagonal through the vertex angles is the angle bisector for both angles.
- Kite Diagonals Theorem: The diagonals of a kite are perpendicular.

To prove that the diagonals are perpendicular, look at $\triangle KET$ and $\triangle KIT$. Both of these triangles are isosceles triangles, which means $\overline{EI}$ is the perpendicular bisector of $\overline{KT}$. Use this information to help you prove the diagonals are perpendicular in the review questions.
YOU TRY!

Find the other two angle measures in the kites below.

a)

b)

c) Use the Pythagorean Theorem to find the length of the sides of the kite.

IMPORTANT TO NOTE: Be careful with the definition of a kite. The congruent pairs are distinct. This means that a rhombus and square cannot be a kite.
3.5e - Kites and Trapezoids in the Coordinate Plane

At this point in the chapter, you should be fluent enough with the methods for working with a quadrilateral in the coordinate plane. Given this, determine the methods to justify that RSTV is a Kite. Simplify all radicals.

YOU TRY!

a) Determine what type of quadrilateral ABCD is.
   \[ A(-3, 3), \; B(1, 5), \; C(4, -1), \; D(1, -5). \] Simplify all radicals.

Hint: If you are only given a set of points when determining what type of quadrilateral a figure is, always plot the points and graph. The visual will help you decide which direction to go.

b) Determine what type of quadrilateral EFGH is.
   \[ E(5, -1), \; F(11, -3), \; G(5, -5), \; H(-1, -3) \]

http://www.mathsisfun.com/geometry/quadrilaterals-interactive.html
Section 3.5 Homework Questions

1. TRAP is an isosceles trapezoid.

   Find:

   \[ \text{a. } m\angle TPA \]
   \[ \text{b. } m\angle PTR \]
   \[ \text{c. } m\angle ZRA \]
   \[ \text{d. } m\angle PZA \]

2. KITE is a kite.

   Find:

   \[ \text{a. } m\angle ETS \]
   \[ \text{b. } m\angle KIT \]
   \[ \text{c. } m\angle IST \]
   \[ \text{d. } m\angle SIT \]
   \[ \text{e. } m\angle ETI \]

3. Can the parallel sides of a trapezoid be congruent? Why or why not?

4. Besides a kite and a rhombus, can you find another quadrilateral with perpendicular diagonals? Explain and draw a picture.

5. Describe how you would draw or construct a kite.
For questions 6-11, find the length of the midsegment or missing side.

6.

7.

8.

9.

10.

11.
For questions 12-17, find the value of the missing variable(s). Simplify all radicals.

12. \( \frac{x + 3}{4x + 7} \)

13. \( \frac{3}{4} \) and \( \frac{8}{x} \)

14. \( \frac{y}{15} \) and \( \frac{2x - 7}{8} \)

15. \( \frac{13y}{125} \) and \( \frac{9y}{(9y)^0} \)

16. \( \frac{112}{(2y + 1)} \) and \( \nu^0 \)
For questions 18-25, determine what type of quadrilateral $ABCD$ is. $ABCD$ could be any quadrilateral that we have learned in this chapter. If it is none of these, write none.

18. $A(1, -2), \ B(7, -5), \ C(4, -8), \ D(-2, -5)$
19. $A(6, 6), \ B(10, 8), \ C(12, 4), \ D(8, 2)$
20. $A(-1, 8), \ B(1,4), \ C(-5, -4), \ D(-5, 6)$
21. $A(5, -1), \ B(9, -4), \ C(6, -10), \ D(3, -5)$
22. $A(-2, 2), \ B(0, 1), \ C'(2, 2), \ D(1, 5)$
23. $A(-7, 4), \ B(-4, 4), \ C(0, 0), \ D(0, -3)$
24. $A(3, 3), \ B(5, -1), \ C(7, 0), \ D(5, 4)$
25. $A(-4, 4), \ B(-1, 2), \ C(2, 4), \ D(-1, 6)$

(need to create integration assignment later)

End of Chapter 3