Chapter 6: The Pythagorean Theorem & Right Triangle

Trigonometry

Right triangles have played a part in the story of humankind extending back to the earliest of our civilizations. As we can see in the picture on the left, the ancient Egyptians not only tried to work with right triangles in building the pyramids but also in explaining the relationships of their gods. The ancient Greeks were obsessed with its properties and relationships – a fact which drove Pythagoras to dedicate a lifetime of work on its calculations and proof! Today, working with right triangles and trigonometry extends through everything from interior design, art, engineering, to understanding outer space!

In this chapter, we will first prove the Pythagorean Theorem and its converse, followed by analyzing the sides of certain types of triangles. Then, we will introduce trigonometry’s tangent, sine and cosine ratios. Finally, we will extend sine and cosine to any triangle, through the Law of Sine and the Law of Cosine.

Research Assignment: Historic Figures in Mathematics

The Greek Mathematicians Pythagoras and Eratosthenese are important figures in human history and for their contributions to mathematics and logical and philosophical thought. To enrich your appreciation for the history of mathematics, research each of these figures and identify their contributions to mathematics and society as a whole. During your research make sure to answer the pre-requisite who, what where, why, and when type questions and provide any additional information you feel would help a reader/viewer better understand these men. After doing your research you need to create a “Prezi” (www.prezi.com), PowerPoint, or a short video to present your findings. Be creative but make sure to answer the essential questions. Please include a proper bibliography. Any additional pictures, diagrams, maps, etc., are encouraged!
**Concepts & Skills**

The concepts and skills covered in this chapter include

- Prove and use the Pythagorean Theorem.
- Identify common Pythagorean triples.
- Use the Pythagorean Theorem to find the area of isosceles triangles.
- Understand the converse of the Pythagorean Theorem.
- Identify acute and obtuse triangles from side measures.

- Identify similar triangles inscribed in a larger triangle.
- Evaluate the geometric mean.
- Find the length of an altitude or leg using the geometric mean.
- Identify and use the ratios involved with isosceles right triangles.
- Identify and use the ratios involved with 30-60-90 triangles.

- Use the tangent, sine and cosine ratios in a right triangle.
- Understand these trigonometric ratios in special right triangles.
- Use a scientific calculator to find sine, cosine and tangent.
- Use trigonometric ratios in real-life situations.
- Use the inverse trigonometric ratios to find an angle in a right triangle.
- Solve a right triangle.
- Apply inverse trigonometric ratios to real-life situation and special right triangles.

- Identify and use the Law of Sine and Cosine.
Section 6.1: The Pythagorean Theorem

INSTANT RECALL:

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. Simplify the radical.
   a. \( \sqrt{50} \)
   b. \( \sqrt{27} \)
   c. \( \sqrt{272} \)
4. Perform the indicated operations on the following numbers. Simplify all radicals.
   a. \( 2\sqrt{10} + \sqrt{160} \)
   b. \( 5\sqrt{6} + 4\sqrt{18} \)
   c. \( \sqrt{8} \cdot 2\sqrt{2} \)

Know What? All televisions dimensions refer to the diagonal of the rectangular viewing area. Therefore, for a 52” TV, 52” is the length of the diagonal. High Definition Televisions (HDTVs) have sides in the ratio of 16:9. What is the length and width of a 52” HDTV? What is the length and width of an HDTV with a \( \frac{y}{y} \) long diagonal?
6.1a - The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but we have never proved it. Recall that the sides of a right triangle are called legs (the sides of the right angle) and the side opposite the right angle is the hypotenuse. For the Pythagorean Theorem, the legs are “a” and “b” and the hypotenuse is “c”.

To get a better appreciation for the theorem, its proof, uses, and history, visit the following website – it is an excellent summary of all things Pythagorean!

https://sites.google.com/site/pythagoreanthreorema2b2c2/home

**Pythagorean Theorem**: Given a right triangle with legs of lengths a and b and a hypotenuse of length c, then, \( a^2 + b^2 = c^2 \).

**Investigation 6.1a: Proof of the Pythagorean Theorem**

**Proof #1**

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in square and a right triangle with legs of 3 and 4 inches.
2. Cut out the triangle and square and arrange them like the picture on the right.

3. This theorem relies on area. Recall from a previous math class, that the area of a square is length times width. But, because the sides are the same you can rewrite this formula as

\[ A_{\text{square}} = \text{length} \times \text{width} = \text{side} \times \text{side} = \text{side}^2. \]

So, the Pythagorean Theorem can be interpreted as

\[ (\text{square with side } a)^2 + (\text{square with side } b)^2 = (\text{square with side } c)^2. \]

In this Investigation, the sides are 3, 4 and 5 inches. What is the area of each square?

4. Now, we know that \( 9 + 16 = 25 \), or \( 3^2 + 4^2 = 5^2 \). Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

**Proof #2**

This proof is “more formal,” meaning that we will use letters, \( a \), \( b \), and \( c \), to represent the sides of the right triangle. In this particular proof, we will take four right triangles, with legs \( a \) and \( b \) and hypotenuse \( c \) and make the areas equal.
For two animated proofs, go to http://www.mathsisfun.com/pythagoras.html and scroll down to “And You Can Prove the Theorem Yourself.”

**6.1b - Using the Pythagorean Theorem**

The Pythagorean Theorem can be used to find a missing side of any right triangle, to prove that three given lengths can form a right triangle, to find Pythagorean Triples, and to find the area of an isosceles triangle. Here are several examples.

**IMPORTANT TO REMEMBER! Fully simplify ALL radicals to their lowest form.**

At this point in your math education you should be comfortable solving a generic right triangle for a missing side. But just in case, let’s review a quick problem below.

Example 1: Find the length of the hypotenuse of the triangle below.

\[ 8^2 + 15^2 = c^2 \]
\[ 64 + 225 = c^2 \]
\[ 289 = c^2 \]
\[ \sqrt{289} = \sqrt{c^2} \]
\[ 17 = c \]

Remember, when you take the square root of an equation, usually the answer is +17 or -17. However, because we are looking for length, we only use the positive answer. *Length is never negative.*
Example 2: Find the missing side of the right triangle below.

\[ 7^2 + b^2 = 14^2 \]
\[ 49 + b^2 = 196 \]
\[ 147 = b^2 \]
\[ \sqrt{147} = \sqrt{b^2} \]
\[ 7\sqrt{3} = b \]

Example 3: What is the diagonal of a rectangle with sides 10 and \(16\sqrt{5}\)?

How should this be solved?

Example 4 – **Apply It!** A 16 foot ladder rests up against the side of a house. The base of the ladder is 4 feet from the base of the house. How far up the house does the ladder rest?

**IMPORTANT**: Draw a diagram of this situation BEFORE you try to solve the problem. This is essential for solving most Pythagorean Theorem and Trigonometry problems.

**Internet & Video Links**

2. [https://www.khanacademy.org/math/geometry/triangles/v/introduction-to-the-pythagorean-theorem](https://www.khanacademy.org/math/geometry/triangles/v/introduction-to-the-pythagorean-theorem)
6.1c - Pythagorean Triples

Draw a right triangle were 8, 15, and 17 are the sides. This combination of numbers is referred to as a *Pythagorean Triple*.

- **Pythagorean Triple**: A set of three whole numbers that makes the Pythagorean Theorem true.

The most frequently used Pythagorean triple is 3, 4, 5. Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Therefore, 6, 8, 10 and 9, 12, 15 are also sides of a right triangle. Other Pythagorean triples are:

\[3, 4, 5 \quad 5, 12, 13 \quad 7, 24, 25 \quad 8, 15, 17\]

There are infinitely many Pythagorean triples. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

**Example 5**: Is 20, 21, 29 a Pythagorean triple?

\[
\begin{align*}
20^2 + 21^2 &= 29^2 \\
400 + 441 &= 841 \\
841 &= 841
\end{align*}
\]

Therefore, because the left side of the equation is equal to the right, these side lengths show that this is a Pythagorean Triple.

**IMPORTANT**: If the left and right sides do not agree, it is NOT a triple!

**You Try!**

The given lengths are two sides of a right triangle. All three side lengths of the triangle are integers and together form a Pythagorean Triple. Find the length of the third side and tell whether it is a leg or a hypotenuse.

a) 24 and 51  
b) 20 and 48  
c) 72 and 75
6.1d Area of an Isosceles Triangle

There are many different applications of the Pythagorean Theorem. One way to use The Pythagorean Theorem is to identify the heights in isosceles triangles so you can calculate the area. The area of a triangle is \( \frac{1}{2} bh \), where \( b \) is the base and \( h \) is the height (or altitude).

\[
\text{Area of a Triangle} = \frac{1}{2} bh
\]

If you are given the base and the sides of an isosceles triangle, you can use the Pythagorean Theorem to calculate the height.

Example 6: What is the area of the isosceles triangle?

Remember, and isosceles triangle can be “cut” into 2 equal halves and each of these halves is a right triangle. That means each right triangle has a base of \( \frac{14}{2} = 7.5 \) and hypotenuse of 9. To find the area of the triangle however, we need its height. This is where you need to use the Pythagorean Theorem to help.

\[
(7.5)^2 + b^2 = 9^2
\]
\[
56.25 + b^2 = 81
\]
\[
24.75 = b^2 \quad \text{This is for the area of one half of the isosceles triangle.}
\]
\[
\sqrt{24.75} = \sqrt{b^2}
\]
\[
4.98 \approx b
\]

To find the full area simply multiply your answer by 2

\[
2 \cdot 4.98 \approx 9.96 \text{ units}^2 \quad \text{Remember, square your units!}
\]
Is there another way to solve the same problem that may not require as much work? Hint: setup the equations!

You try!

Use Δ STU as the base diagram to help you solve each of the following.

a) If ST = 18ft and SU = 30ft, find the area of the triangle.

b) If TU = 26m and SU = 20m, find the area of the triangle.

c) If the height of the triangle is 14cm, and SU = 96cm, find the area of the triangle.

Internet & Video Links

1. [https://www.khanacademy.org/math/geometry/triangles/v/triangle-area-proofs](https://www.khanacademy.org/math/geometry/triangles/v/triangle-area-proofs)

2. [https://www.khanacademy.org/math/geometry/similarity/triangle_similarity/v/similarity-example-where-same-side-plays-different-roles](https://www.khanacademy.org/math/geometry/similarity/triangle_similarity/v/similarity-example-where-same-side-plays-different-roles)
Section 6.1 Homework Problems

Find the length of the missing side. Simplify all radicals.

1. 

2. 

3. 

4. 

5. 

6. 
7. If the legs of a right triangle are 10 and 24, then the hypotenuse is ______________.

8. If the sides of a rectangle are 12 and 15, then the diagonal is ______________.

9. If the legs of a right triangle are $x$ and $y$, then the hypotenuse is ______________.

10. If the sides of a square are 9, then the diagonal is ______________.

Determine if the following sets of numbers are Pythagorean Triples.

11. 12, 35, 37
12. 9, 17, 18
13. 10, 15, 21
14. 11, 60, 61
15. 15, 20, 25
16. 18, 73, 75

Find the area of each triangle below. Simplify all radicals.

17. [Diagram of a triangle with sides 16 and 20]
Find the length between each pair of points.

20. (-1, 6) and (7, 2)
21. (10, -3) and (-12, -6)
22. (1, 3) and (-8, 16)

23. What are the length and width of a 42" HDTV? Round your answer to the nearest tenth.

24. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42" Standard definition TV? Round your answer to the nearest tenth.

25. Challenge An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are $s$, find the area, using the technique learned in this section. Leave your answer in simplest radical form.

26. Find the area of an equilateral triangle with sides of length 8.
27. *Research Pythagorean Theorem Proofs*: One Pythagorean proof is attributed to former US president James A Garfield. Research this proof and write an explanation for how he proved the theorem. Diagrams are appropriate.

28. In baseball, the distance of the path between each base is 90 feet and the paths form right angles. How far does the need to travel if it is thrown from home plate directly to second base? Draw a diagram to help.

29. You tie a balloon to a stake in the ground. The rope is 10 feet long. As the wind picks up, you observe that the balloon is now 6 feet away from the stake. How far above the ground is the balloon now?

30. Solve for $x$ if the lengths of the two legs of a right triangle are $2x$ and $2x+4$, and the length of the hypotenuse is $4x-4$. (Hint: quadratic)
Section 6.2:
Converse of the Pythagorean Theorem

6.2a Converse of the Pythagorean Theorem

In the last lesson, you learned about the Pythagorean Theorem and how it can be used. The converse of the Pythagorean Theorem is also true.

Pythagorean Theorem Converse: If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle’s angle measurements.

Example 1: Determine if the triangle below is a right triangle.

To use the converse theorem, set up the Pythagorean Theorem with the information given in the problem and see if the left side of the equation is equal to the right. If they are, the triangle is a right triangle.

\[16^2 + 8^2 = (8\sqrt{5})^2\]
\[256 + 64 = 320\]
\[320 = 320\]

Therefore the triangle in the diagram is a right triangle.

YOU TRY!

Use the information in the diagram below to determine is the triangle is a right triangle.
6.2b Identifying Acute and Obtuse Triangles

We can extend the converse of the Pythagorean Theorem to determine if a triangle has an obtuse angle or is acute. We know that if the sum of the squares of the two smaller sides equals the square of the larger side, then the triangle is right. We can also interpret the outcome if the sum of the squares of the smaller sides does not equal the square of the third.

- **Theorem #1**: If the sum of the squares of the two shorter sides in a right triangle is greater than the square of the longest side, then the triangle is **acute**.
- **Theorem #2**: If the sum of the squares of the two shorter sides in a right triangle is less than the square of the longest side, then the triangle is **obtuse**.

In other words: The sides of a triangle are a, b, and c, and c > b and c > a.

If \( a^2 + b^2 > c^2 \), then the triangle is acute.

If \( a^2 + b^2 = c^2 \), then the triangle is right.

If \( a^2 + b^2 < c^2 \), then the triangle is obtuse.

**Proof of Theorem #1**

**Given:** In \( \triangle ABC \), \( a^2 + b^2 > c^2 \), where \( c \) is the longest side.

In \( \triangle LMN \), \( \angle N \) is a right angle.

![Diagram of \( \triangle ABC \) and \( \triangle LMN \)](diagram.png)

**Prove:** \( \triangle ABC \) is an acute triangle. (all angles are less than 90°)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. In ( \triangle ABC ), ( a^2 + b^2 &gt; c^2 ), and ( c ) is the longest side. In ( \triangle LMN ), ( \angle N ) is a right angle.</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( a^2 + b^2 = h^2 )</td>
<td>Pythagorean Theorem</td>
</tr>
</tbody>
</table>
Statement

3. \( c^2 < h^2 \)

4. \( c < h \)

5. \( \angle C \) is the largest angle in \( \triangle ABC \).

6. \( m\angle N = 90^\circ \)

7. \( m\angle C < m\angle N \)

8. \( m\angle C < 90^\circ \)

9. \( \angle C \) is an acute angle.

10. \( \triangle ABC \) is an acute triangle.

Reason

Transitive PoE

Take the square root of both sides

The largest angle is opposite the longest side.

Definition of a right angle

SSS Inequality Theorem

Transitive PoE

Definition of an acute angle

If the largest angle is less than \( 90^\circ \), then all the angles are less than \( 90^\circ \).

The proof of Theorem #2 is very similar and is in the review questions.

YOU TRY!

Determine if the following triangles are acute, right or obtuse.

a)

b)

c) Graph \( A(-4, 1), B(3, 8), \) and \( C(9, 6) \). Determine if \( \triangle ABC \) is acute, obtuse, or right.
Section 6.2 Homework Questions

1. The two shorter sides of a triangle are 9 and 12.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
2. The two longer sides of a triangle are 24 and 25.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
3. The lengths of the sides of a triangle are $8x$, $15x$, and $17x$. Determine if the triangle is acute, right, or obtuse.

Determine if the following lengths make a right triangle.

4. 15, 20, 25
5. 20, 25, 30
6. $8\sqrt{3}, 6, 2\sqrt{39}$

Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76
Graph each set of points and determine if \( \triangle ABC \) is acute, right, or obtuse.

16. \( A(3, -5), B(-5, -8), C(-2, 7) \)
17. \( A(5, 3), B(2, -7), C(-1, 5) \)
18. **Writing** Explain the two different ways you can show that a triangle in the coordinate plane is a right triangle.

The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All sides are perpendicular.

19. Find \( c \).
20. Find \( d \).

21. **Writing** Explain why \( m\angle A = 90^\circ \).
Section 6.3: Using Similar Right Triangles

INSTANT RECALL:

1. If two triangles are right triangles, does that mean they are similar? Explain.
2. If two triangles are isosceles right triangles, does that mean they are similar? Explain.
3. Solve the ratio: \( \frac{3}{x} = \frac{x}{27} \)
4. If the legs of an isosceles right triangle are 4, find the length of the hypotenuse. Draw a picture and simplify the radical.

6.3a - Inscribed Similar Triangles

You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length. Let’s look at a right triangle, with an altitude drawn from the right angle.

There are three right triangles in this picture, \( \triangle ADB, \triangle DCA, \) and \( \triangle CBA \). Both of the two smaller triangles are similar to the larger triangle because they each share an angle with \( \triangle ADB \). That means all three triangles are similar to each other.

- Theorem #3: If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.

The proof of Theorem #3 is in the review questions.
Example 1: Write the similarity statement for the triangles below.

Applying theorem #3, we see that the altitude ST is drawn from the right angle R. Therefore the inscribed (inner) triangles must be similar.

\[ \triangle RIT \sim \triangle RET \sim \triangle IRE \]

Remember the symbol \( \sim \) means similarity.

We can also use the side proportions to find the length of the altitude.

Example 2: Find the value of \( x \).

Based on the information in the diagram we know that Theorem #3 applies so we can solve for \( x \) by setting up proportions.

\[ \frac{x}{6} = \frac{10}{8} \]

and through cross multiplication we get \( 8x=60 \) or \( x=7.5 \)

YOU TRY!

Find the value of \( x \) and \( y \) in the diagram below.
6.3b: The Geometric Mean

You are probably familiar with the arithmetic mean, which *divides the sum* of a number of items by the number of items. This is commonly used to determine the average test score for a group of students.

The geometric mean is a different sort of average, which takes the *nth root of the product* of *n* numbers. In this text, we will primarily compare two numbers, so we would be taking the square root of the product of two numbers. This mean is commonly used with rates of increase or decrease.

| Geometric Mean: The geometric mean of two positive numbers *a* and *b* is the number *x*, such that \( \frac{a}{x} = \frac{x}{b} \) or \( x^2 = ab \) and \( x = \sqrt{ab} \). |

*Example 5*: Find the geometric mean of 24 and 36

Based on the theorem above, the geometric mean = \( \sqrt{(24)(36)} = 12\sqrt{6} \)

**YOU TRY!**

A) Find the geometric mean of 18 and 54.
B) Find the geometric mean of 72 and 11.
C) Find the geometric mean of 11.6 and 28.9

The same principle can be used to find the altitude of a right triangle.

*Example 6*: Find the value of \( x \).

- Theorem #4: In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of the altitude is the geometric mean of these two segments.
- Theorem #5: In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments.
The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.

In other words

Theorem #4: \( \frac{BC}{AC} = \frac{AC}{DC} \) so for the problem \( \frac{9}{x} = \frac{x}{27} \) where \( x = \sqrt{9 \times 27} = 9\sqrt{3} \)

Similarly, we can find the length of the sides AB and AD by

Theorem #5: \( \frac{BC}{AB} = \frac{AB}{DB} \) and \( \frac{DC}{AD} = \frac{AD}{DB} \)

So for the problem \( AB = \frac{9}{x} = \frac{x}{36} \) where \( x = \sqrt{9 \times 36} = 18 \)

And for \( AD = \frac{27}{x} = \frac{x}{36} \) where \( x = \sqrt{27 \times 36} = 18\sqrt{3} \)

**YOU TRY!**

Find the value of \( x \) and \( y \).
Section 6.3 Homework Questions

Use the diagram to answer questions 1-4.

1. Write the similarity statement for the three triangles in the diagram.
2. If $JM = 12$ and $ML = 9$, find $KM$.
3. Find $JK$.
4. Find $KL$.

Find the geometric mean between the following two numbers. Simplify all radicals.

5. 16 and 32
6. 45 and 35
7. 10 and 14
8. 28 and 42
9. 40 and 100
10. 51 and 8

Find the length of the missing variable(s). Simplify all radicals.

11. 

12. 
22. Last year Poorva’s rent increased by 5% and this year her landlord wanted to raise her rent by 7.5%. What is the average rate at which her landlord has raised her rent over the course of these two years?

23. Mrs. Smith teaches AP Calculus. Between the first and second years she taught the course her students’ average score improved by 12%. Between the second and third years, the scores increased by 9%. What is the average rate of improvement in her students’ scores?

24. According to the US Census Bureau, http://www.census.gov/ipc/www/idb/country.php the rate of growth of the US population was 0.8% and in 2009 it was 1.0%. What was the average rate of population growth during that time period?

**Algebra Connection** A geometric sequence is a sequence of numbers in which each successive term is determined by multiplying the previous term by the common ratio. An example is the sequence 1, 3, 9, 27. Here each term is multiplied by 3 to get the next term in the sequence. Another way to look at this sequence is to compare the ratios of the consecutive terms.

25. Find the ratio of the 2\textsuperscript{nd} to 1\textsuperscript{st} terms and the ratio of the 3\textsuperscript{rd} to 2\textsuperscript{nd} terms. What do you notice? Is this true for the next set (4\textsuperscript{th} to 3\textsuperscript{rd} terms)?

26. Given the sequence 4, 8, 16,..., if we equate the ratios of the consecutive terms we get: \( \frac{8}{4} = \frac{16}{8} \). This means that 8 is the \underline{__________} of 4 and 16. We can generalize this to say that every term in a geometric sequence is the \underline{__________} of the previous and subsequent terms.
Section 6.4: Special Right Triangles

INSTANT RECALL:

Find the value of the missing variable(s). Simplify all radicals.

1. 

2. 

3. 

4. Do the lengths 6, 6, and $6\sqrt{2}$ make a right triangle?

5. Do the lengths $3, 3\sqrt{3}$ and 6 make a right triangle?
Know What? The Great Giza Pyramid is a pyramid with a square base and four isosceles triangles that meet at a point. It is thought that the original height was 146.5 meters and the base edges were 230 meters.

First, find the length of the edge of the isosceles triangles. Then, determine if the isosceles triangles are also equilateral triangles. Round your answers to the nearest tenth.

You can assume that the height of the pyramid is from the center of the square base and is a vertical line.

6.4a - Isosceles Right Triangles

There are two types of special right triangles, based on their angle measures. The first, as we saw earlier is an isosceles right triangle. However, when we looked at them before we only really paid attention to the side lengths and not the angles. We are now beginning our transition to working with the angles of right triangles. This will culminate in our introduction to right triangle trigonometry. The relationships we establish in this section are essential for working with trigonometric problems in this course and future courses you will take. These special angles are the basis of many of the trigonometric relations you will work with. You need to commit them to memory!
As we studied before, in an isosceles triangle, the legs (2 sides) are congruent and then by the Base Angles Theorem, the base angles will also be congruent. Therefore, if the non-base angle is known to be 90°, then is the triangle sum theorem holds, the measure of the base angles MUST be 45° each. We call this a 45-45-90 triangle.

![Diagram of a 45-45-90 triangle]

Each angle is 45° and the two legs are congruent

**IMPORTANT TO REMEMBER:** Because the three angles are always the same, all isosceles right triangles are similar.

If we draw a random isosceles triangle and label the length of the legs $x$ and the length of the hypotenuse $h$, we can use the Pythagorean Theorem to establish a relationship for the legs to the hypotenuse. This is shown in the diagram below.

![Diagram showing the Pythagorean Theorem]

$$x^2 + x^2 = h^2$$

$$2x^2 = h^2$$

$$x\sqrt{2} = h$$

**45-45-90 Corollary:** If a triangle is an isosceles right triangle, then its sides are in the extended ratio $x:x:x\sqrt{2}$.

So, anytime you have a right triangle with congruent legs or congruent angles, then the sides will always be in the ratio $x:x:x\sqrt{2}$. The hypotenuse is
always \( x\sqrt{2} \) because that is the longest length. This is a specific case of the Pythagorean Theorem, so it will still work, if for some reason you forget this corollary.

In the following example we will show that both ways of solving will work – but clearly, if you remember the relationships, one is much easier and quicker than the other!

**Example**: Find the length of the missing sides.

Method #1: Since we see the right triangle is isosceles and we know a side length, based on the 45-45-90 corollary, the hypotenuse must be \( 6\sqrt{2} \).

Method #2: If you don’t remember the corollary, just use the Pythagorean Theorem!

\[
6^2 + 6^2 = c^2 \\
36 + 36 = c^2 \\
72 = c^2 \\
\sqrt{72} = \sqrt{c^2} \\
c = 6\sqrt{2}
\]

Same result, but much more work!

**Internet & Video Links**

YOU TRY!

Work the problems below to practice the 45-45-90 corollary. Notice, in b and c you are given the hypotenuse and no leg length! Set up an equation using the corollary $x\sqrt{2}$ to help you with these problems.

![Diagram of 45-45-90 triangle with sides 9, 9, and 9√2]

The second special right triangle is called a 30-60-90 triangle, after the three angles that come from cutting an equilateral triangle in half. Remember, equilateral triangles have three equal sides and three equal 60° angles. When you cut it in half we get the diagram below.

![Diagram of 30-60-90 triangle with sides 2, 1, and 2√3]

*The top angles are each 30° and the shorter leg is 1 in because the altitude of an equilateral triangle is also the angle and perpendicular bisector.*
If we assume the length of the base is $x$ and the hypotenuse is $2x$, using the Pythagorean Theorem to find the longer leg (altitude) we get the side length of $x\sqrt{3}$.

**30-60-90 Corollary**: If a triangle is a 30-60-90 triangle, then its sides are in the extended ratio $x : x\sqrt{3} : 2x$.

The use of the Pythagorean Theorem to find the side lengths proves the 30-60-90 Corollary. The shortest leg is always $x$, the longest leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. As with the 45-45-90 corollary, if you ever forget this corollary, then you can still use the Pythagorean Theorem to find the lengths.

*Example*: Find the length of the missing sides.

![Diagram](image)

NOTE: This 30-60-90 triangle is lying in its side but 5 is still considered the base!

**Method #1**: Using the facts from the corollary and given 5 is the base, the hypotenuse must be 10 ($2x = 2(5) = 10$) and the long leg must be $5\sqrt{3}$. 

32
Method #2: Use the Pythagorean Theorem. You still need to at least recognize that the hypotenuse is 2x or 10 or you do not have enough information to use the theorem. If you go through the math you get

\[5^2 + b^2 = 10^2\]
\[25 + b^2 = 100\]
\[75 = b^2\]
\[\sqrt{75} = \sqrt{b^2}\]
\[b = 5\sqrt{3}\]

Again, same result but a bit more work!

YOU TRY! Find the missing side lengths in the diagrams below.

a) 

b) 

c) 

Challenge: Find the measure of x in the diagram below. Think special angles!
Internet & Video Links


Know What? Revisited The line that the vertical height is perpendicular to is the diagonal of the square base. This length (blue) is the same as the hypotenuse of an isosceles right triangle because half of a square is an isosceles right triangle. So, the diagonal is \(230\sqrt{2}\). Therefore, the base of the right triangle with 146.5 as the leg is half of \(230\sqrt{2}\) or \(115\sqrt{2}\). Do the Pythagorean Theorem to find the edge.

\[
edge = \sqrt{\left(115\sqrt{2}\right)^2 + 146.5^2} \approx 218.9 \text{ m}
\]

In order for the sides to be equilateral triangles, this length should be 230 meters. It is not, so the triangles are isosceles.
Section 6.4 Homework Questions

1. In an isosceles right triangle, if a leg is $x$, then the hypotenuse is __________.
2. In a 30-60-90 triangle, if the shorter leg is $x$, then the longer leg is __________ and the hypotenuse is __________.
3. A square has sides of length 15. What is the length of the diagonal?
4. A square’s diagonal is 22. What is the length of each side?
5. A rectangle has sides of length 4 and $4\sqrt{3}$. What is the length of the diagonal?
6. A baseball diamond is a square with 90 foot sides. What is the distance from home base to second base? (HINT: It’s the length of the diagonal).

For questions 7-18, find the lengths of the missing sides.

7.

8.

9.
10. \[ h \quad 10 \quad 30^\circ \]

11. \[ j \quad k \]

12. \[ x \quad y \]

13. \[ n \]

14. \[ p \quad q \]

15. \[ s \]

16. Do the lengths $18$, $x$, and $16\sqrt{2}$ make a special right triangle? If so, which one?

17. Do the lengths $27$, $a$, and $b$ make a special right triangle? If so, which one?

18. Do the lengths $12\sqrt{5}$, $p$, and $q$ make a special right triangle? If so, which one?

19. Do the lengths $8\sqrt{2}$, $8\sqrt{6}$, and $16\sqrt{2}$ make a special right triangle? If so, which one?

20. Do the lengths $4\sqrt{3}$, $4\sqrt{6}$, and $8\sqrt{3}$ make a special right triangle? If so, which one?

21. Find the measure of $x$. 
22. Find the measure of $y$.

23. What is the ratio of the sides of a rectangle if the diagonal divides the rectangle into two 30-60-90 triangles?

24. What is the length of the sides of a square with diagonal 8 in?

For questions 25-28, it might be helpful to recall #25 from section 8.1.

25. What is the height of an equilateral triangle with sides of length 3 in?

26. What is the area of an equilateral triangle with sides of length 5 ft?

27. A regular hexagon has sides of length 3 in. What is the area of the hexagon? *(Hint: the hexagon is made up of 6 equilateral triangles.)*

28. The area of an equilateral triangle is $36\sqrt{3}$. What is the length of a side?

29. If a road has a grade of $30^\circ$, this means that its angle of elevation is $30^\circ$. If you travel 1.5 miles on this road, how much elevation have you gained in feet (5280 ft = 1 mile)?

30. Four isosceles triangles are formed when both diagonals are drawn in a square. If the length of each side in the square is $s$, what are the lengths of the legs of the isosceles triangles?
Section 6.5: Tangent, Sine and Cosine

INSTANT RECALL:

1. The legs of an isosceles right triangle have length 14. What is the hypotenuse?
2. Do the lengths 8, 16, 20 make a right triangle? If not, is the triangle obtuse or acute?
3. In a 30-60-90 triangle, what do the 30, 60, and 90 refer to?
4. Find the measure of the missing lengths.

6.5a - What is Trigonometry?

The word trigonometry comes from two words meaning triangle and measure. In this lesson we will define the three primary trigonometric (or trig) relations. In truth, there are 6 relations but the remaining three are the inverse of the primary relations so we will only cover the primary at this time. You will see the inverse relations in Integrated Math 3.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

In trigonometry, sides are named in reference to a particular angle. We typically use the Greek letter θ, theta, to denote the angle that we are
working with. However, the Greek letters α, alpha, and β, beta, are also used to label the angles.

The hypotenuse of a triangle is always the same (the longest side), but the terms adjacent and opposite sides are used to define the sides with reference to the angle Θ. **A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle.**

In the diagram above, the opposite and adjacent sides are in reference to angle A (alpha). However if we reference angle B (beta) line BC becomes the adjacent side and line CA become the opposite.

**IMPORTANT TO REMEMBER:** We never reference the right angle when referring to trig ratios.
Example: Identify the adjacent and opposite sides when the reference angle is A.

Based on the diagram the adjacent side is 35 (AB) while the opposite side is 12 (BC)

Using the diagram above, what are the adjacent and opposite sides if the reference angle is B?

6.5b Sine, Cosine, and Tangent Ratios

The three basic trig ratios are called, sine, cosine and tangent. At this point, we will only take the sine, cosine and tangent of acute angles. However, you will learn that you can use these ratios with obtuse angles as well.

Sine Ratio: For an acute angle θ, in a right triangle, the sin θ, is equal to the ratio of the opposite side over the hypotenuse of the triangle.
Using the triangle above, \( \sin A = \frac{a}{c} = \frac{\text{opposite}}{\text{hypotenuse}} \), when A is the reference angle.

When B is the reference angle the ratio is \( \sin B = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}} \).

EXAMPLE: Find the sine of angle X in the diagram below

Since x is the reference angle, the adjacent side is XY and the opposite is ZY. However, to find the Sinx (sine of angle x) we use the ratio established above so we have

\[
\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{20}{29} \approx 0.6897
\]

What does this mean? Literally it is the ratio of the opposite side to the hypotenuse so we know that based on this right triangle, the opposite side is 68.97% the length of the hypotenuse. This ratio will equate to an angle but for right now let’s leave it as a side ratio.

**Cosine Ratio:** For an acute angle \( \Theta \) in a right triangle; the \( \cos \Theta \) is equal to the **ratio of the adjacent side over the hypotenuse** of the triangle.
Using the triangle above, \( \cos A = \frac{b}{c} = \frac{\text{adjacent}}{\text{hypotenuse}} \) when A is the reference angle

and \( \cos B = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}} \) when B is the reference angle.

EXAMPLE: Find the cosine of angle X in the diagram below

Since x is the reference angle, the adjacent side is XY and the opposite is ZY. However, to find the \( \cos x \) (cosine of angle x) we use the ratio established above so we have

\[
\cos x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{21}{29} \approx 0.7241
\]

What does this mean? Literally it is the ratio of the adjacent side to the hypotenuse so we know that based on this right triangle, the adjacent side is 72.41% the length of the hypotenuse. This ratio will equate to an angle but for right now let’s leave it as a side ratio.

**Tangent Ratio**: For an acute angle \( \theta \), in a right triangle, the \( \tan \theta \) is equal to the **ratio of the opposite side over the adjacent side**. Tangent can also be thought of as the SLOPE of the triangle as it is the ratio of the rise over the run:

\[
\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \text{slope}
\]
Using the triangle above, \( \tan A = \frac{a}{b} = \frac{\text{opposite}}{\text{adjacent}} \) when A is the reference angle

and \( \tan B = \frac{b}{a} = \frac{\text{opposite}}{\text{adjacent}} \) when B is the reference angle.

EXAMPLE: Find the tangent of angle X in the diagram below

Since x is the reference angle, the adjacent side is XY and the opposite is ZY. However, to find the \( \tan x \) (tangent of angle x) we use the ratio established above so we have

\[
\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{20}{21} \approx 0.9524
\]

What does this mean? Literally it is the ratio of the opposite side to the adjacent side so we know that based on this right triangle, the opposite side is 95.24% the length of the adjacent. This ratio will equate to an angle but for right now let’s leave it as a side ratio.

An easy way to remember ratios is to use the pneumonic SOH-CAH-TOA.

\[
\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}
\]
Internet & Video Links

1. https://www.khanacademy.org/math/trigonometry/basic-trigonometry/basic_trig_ratios/v/basic-trigonometry

YOU TRY!

a) Find the sine, cosine and tangent ratios of $\angle A$.

b) Find the sine, cosine, and tangent of $\angle B$.

A few important points to remember:

- Use the Pythagorean Theorem to find the missing side (if there is one).
- Always reduce ratios to 4 decimal places when a ratio does not terminate. Remember, these are irrational numbers so our answer is an approximation most of the time.
- The values for sine and cosine will only be $0 \leq x \leq 1$ when dealing with right triangles. In future courses you will see that these values can also be negative.
- The tangent ratio can be bigger than 1.
- If two right triangles are similar, then their sine, cosine, and tangent ratios will be the same (because they will reduce to the same ratio).
- If there is a radical in the denominator, rationalize the denominator.
6.5c Sine, Cosine, and Tangent with a Calculator

We now know that the trigonometric ratios are dependent on the ratio of an right triangles sides. Therefore, there is one fixed value for every angle, from 0° to 90°. In days before calculators, math mathematicians and students alike used “trig” tables to look up the value of the ratios for a given angle. Look at the trig table below

<table>
<thead>
<tr>
<th>Angle</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.175</td>
<td>0.9998</td>
<td>0.0316</td>
</tr>
<tr>
<td>2</td>
<td>0.349</td>
<td>0.9994</td>
<td>0.0349</td>
</tr>
<tr>
<td>3</td>
<td>0.523</td>
<td>0.9986</td>
<td>0.0354</td>
</tr>
<tr>
<td>4</td>
<td>0.698</td>
<td>0.9976</td>
<td>0.0369</td>
</tr>
<tr>
<td>5</td>
<td>0.877</td>
<td>0.9962</td>
<td>0.0375</td>
</tr>
<tr>
<td>6</td>
<td>1.045</td>
<td>0.9945</td>
<td>0.1013</td>
</tr>
<tr>
<td>7</td>
<td>1.213</td>
<td>0.9925</td>
<td>0.1728</td>
</tr>
<tr>
<td>8</td>
<td>1.379</td>
<td>0.9903</td>
<td>0.2405</td>
</tr>
<tr>
<td>9</td>
<td>1.546</td>
<td>0.9877</td>
<td>0.3058</td>
</tr>
<tr>
<td>10</td>
<td>1.713</td>
<td>0.9848</td>
<td>0.3673</td>
</tr>
<tr>
<td>11</td>
<td>1.879</td>
<td>0.9816</td>
<td>0.4244</td>
</tr>
<tr>
<td>12</td>
<td>2.045</td>
<td>0.9781</td>
<td>0.4771</td>
</tr>
<tr>
<td>13</td>
<td>2.211</td>
<td>0.9744</td>
<td>0.5254</td>
</tr>
<tr>
<td>14</td>
<td>2.377</td>
<td>0.9703</td>
<td>0.5687</td>
</tr>
<tr>
<td>15</td>
<td>2.543</td>
<td>0.9659</td>
<td>0.6072</td>
</tr>
<tr>
<td>16</td>
<td>2.709</td>
<td>0.9613</td>
<td>0.6416</td>
</tr>
<tr>
<td>17</td>
<td>2.875</td>
<td>0.9563</td>
<td>0.6697</td>
</tr>
<tr>
<td>18</td>
<td>3.041</td>
<td>0.9511</td>
<td>0.6926</td>
</tr>
<tr>
<td>19</td>
<td>3.207</td>
<td>0.9455</td>
<td>0.7105</td>
</tr>
<tr>
<td>20</td>
<td>3.373</td>
<td>0.9397</td>
<td>0.7234</td>
</tr>
<tr>
<td>21</td>
<td>3.539</td>
<td>0.9336</td>
<td>0.7319</td>
</tr>
<tr>
<td>22</td>
<td>3.705</td>
<td>0.9272</td>
<td>0.7357</td>
</tr>
<tr>
<td>23</td>
<td>3.871</td>
<td>0.9205</td>
<td>0.7358</td>
</tr>
<tr>
<td>24</td>
<td>4.037</td>
<td>0.9135</td>
<td>0.7318</td>
</tr>
<tr>
<td>25</td>
<td>4.203</td>
<td>0.9055</td>
<td>0.7254</td>
</tr>
<tr>
<td>26</td>
<td>4.369</td>
<td>0.8972</td>
<td>0.7163</td>
</tr>
<tr>
<td>27</td>
<td>4.535</td>
<td>0.8889</td>
<td>0.7048</td>
</tr>
<tr>
<td>28</td>
<td>4.701</td>
<td>0.8805</td>
<td>0.6909</td>
</tr>
<tr>
<td>29</td>
<td>4.867</td>
<td>0.8718</td>
<td>0.6755</td>
</tr>
<tr>
<td>30</td>
<td>5.033</td>
<td>0.8629</td>
<td>0.6587</td>
</tr>
</tbody>
</table>

As you see, every angle from 0° to 90° is listed with its corresponding sine, cosine, and tangent values.

What do you notice about the values for sine and cosine as the degree of the angles moves from least-to-greatest? Do the values of the two trig functions every equal one-another? If so, at what value and why do you think this is the case?

From time-to-time you may come across a course that requires the use of the tables but for the most part we have moved beyond the use of trig
tables in place of the calculator. Your calculator essentially has the trig tables stored in it. The key advantage of the calculator over the trig tables is that the calculator will give us precise values for decimal angles (45.7°) where as a table will not.

Your TI Nspire has a specific trig button on the left side of the keypad (next to the = sign). When you click on this button, it will show you the primary trig functions we have discussed here plus the three remaining relations and their inverses.

what you see when you click the trig button!
To use the function simply click “sin”, “cos”, or “tan” key and the function will appear in your workflow. To find a trig value for an angle you can enter it two ways but it will give you two different results. **For our uses right now we want to find decimal values of the functions so please use the first method of entering the information in your calculator.**

**IMPORTANT TO NOTE:** Before using trig functions on your Inspire, you need to change the settings to DEGREES. It is automatically set to radians. Radians are a measure you will learn in Integrated Math 3!

“angle” needs to be Degrees! Remember to hit “Make Default” to save the setting!

**To enter the value you MUST put a decimal point after the angle. For the sine of 30° you need to enter sin(30.) in order for it to return the value (0.5). If you do not enter the decimal point it will not give you the value!**
Use these to find the sine, cosine, and tangent of any acute angle.

You can also enter the value with a degree symbol. The big difference however is that the value it returns for you will be the exact trig value – many of which are irrational (radical) numbers. In future courses, this is the preferred value to work with but for now, let’s stick with the decimal values. If you want to work with exact values however, by all means do!

Notice the difference in the value given you depending on how you enter it! They are BOTH the SAME value!

It is really important that you know how and are comfortable using the calculator to find trig values!
YOU TRY!

Find the indicated trigonometric value, using your calculator.

a) \( \sin 78^\circ \)

b) \( \cos 60^\circ \)

c) \( \tan 15^\circ \)

d) What are the values for sine and cosine for 0°? 90°? What do you notice?

e) Find the three trig values for each of the special angles (30°, 45°, 60°)

You should commit these to memory!

<table>
<thead>
<tr>
<th>Function</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6.5d Finding the Sides of a Triangle using Trig Ratios

In the previous section we saw how we can express the relationship of one side of a right triangle to another through the use of a trig ratio. However, let’s not forget, trigonometry is also about angles! In this section we will look at how to use an angle and a side along with a trig ratio to find the missing side of a right triangle.

You may be asking yourself, why can’t I just use the Pythagorean Theorem to find the missing side of a right triangle? That is a logical question, but how useful is the Pythagorean Theorem if you are only given one side? Of course, just knowing one side length is still not enough to find another missing side for every three relationship requires the knowledge of two pieces of information to help work things out! One application of the
trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side.

**IMPORTANT TO REMEMBER:** TRIG RATIOS ARE FORMULAS! They can help you find missing values in a right triangle as long as you set the equation up.

**Example: Using Sine to find a missing side:** You need to be observant to understand where to start with this problem. We see that we are given a 53° angle and the side OPPOSITE the angle is the side we are solving for. We also know that the hypotenuse is 17 cm long. At this point you need to ask yourself, **“What trig ratio involves the opposite side and the hypotenuse?”** The answer, the sine ratio!

\[
\begin{align*}
\text{Sine } \theta &= \frac{\text{opp}}{\text{hyp}} \\
\sin 53^\circ &= \frac{x}{17} \\
17(\sin 53^\circ) &= x \\
x &\approx 13.58\text{ cm}
\end{align*}
\]

There are several important things to point out in the example above.

1. We used the trig ratio for sine as a formula.
2. We solved this equation by eliminating the fraction (multiplied both sides by 17)
3. You need to use your calculator (or trig table) to find the value for the sine of 53°.
4. Your final answer is an approximation because the sine value is irrational.
You Try!  Use the sine function to find the missing side.

Example: Using Cosine to find a missing side: We see that we are given a \(37^\circ\) angle and the side adjacent the angle is the side we are solving for. We also know that the hypotenuse is 13 cm long. At this point you need to ask yourself, "What trig ratio involves the adjacent side and the hypotenuse?"  The answer, the cosine ratio!

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}}
\]

\[
\cos 37^\circ = \frac{x}{13}
\]

\[
13(\cos 37^\circ) = x
\]

\[
x \approx 10.38cm
\]

There are several important things to point out in the example above.

1. We used the trig ratio for cosine as a formula.
2. We solved this equation by eliminating the fraction (multiplied both sides by 13)
3. You need to use your calculator (or trig table) to find the value for the cosine of \(37^\circ\).
4. Your final answer is an approximation because the cosine value is irrational.
You Try!  Use the cosine function to find the missing side.

\[
\text{a) } \quad \cos 73° = \frac{14}{x} \\
\text{b) } \quad \cos 64° = \frac{11}{x}
\]

IMPORTANT TO NOTE: In both of the previous examples, if you were given the opposite and adjacent sides for the respective problems, you would still use the same ratios to find the missing hypotenuse!

**Example: Using Tangent to find a missing side:** We see that we are given a 39° angle and the side adjacent the angle is the side we are solving for. We also know that the opposite is 17 cm long. At this point you need to ask yourself, "**What trig ratio involves the adjacent and opposite sides?**" The answer, the tangent ratio!

\[
\tan 39° = \frac{17}{x}
\]

\[
x \approx 21 \text{ cm}
\]

There are several important things to point out in the example above.

1. We used the trig ratio for tangent as a formula.
2. We solved this equation by eliminating the fraction (multiplied both sides by x) and then dividing both sides by \( \tan 39° \)
3. You need to use your calculator (or trig table) to find the value for the tangent of 39°.
4. Your final answer is an approximation because the tangent value is irrational.
**You Try!** Use the tangent function to find the missing side.

![Triangle Diagram](image1)

b)

**IMPORTANT TO NOTE:** Once you know 2 sides, you can use the Pythagorean Theorem to find the remaining missing side if asked!

**YOU TRY!** These are a bit more challenging but use your knowledge of the trig ratios to help you find both of the missing sides. (**Hint:** you have more than one option as to where to start!)

![Triangle Diagram](image2)

b)

![Triangle Diagram](image3)

c)

**Internet & Video Links**

1. [https://www.khanacademy.org/math/trigonometry/basic-trigonometry/basic_trig_ratios/v/basic-trigonometry-ii](https://www.khanacademy.org/math/trigonometry/basic-trigonometry/basic_trig_ratios/v/basic-trigonometry-ii)
6.5e Angles of Depression and Elevation

Another practical application of the trigonometric functions is to find the measure of lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

- **Angle of Depression**: The angle measured from the horizon or horizontal line, down.

  ![Diagram of Angle of Depression]

- **Angle of Elevation**: The angle measure from the horizon or horizontal line, up.

  ![Diagram of Angle of Elevation]

**EXAMPLE**: A man who is 2m tall stands 30m from the base of a tree. The man looks up at the top of the tree with an angle of elevation of 28°. How tall is the tree?
Let the height of the tree be $h$. Sketch a diagram to represent the situation.

\[
\tan 28^\circ = \frac{(h - 2)}{30}
\]

\[
h - 2 = 30 \tan 28^\circ
\]

\[
h = (30 \cdot 0.5317) + 2 \leftarrow \tan 28^\circ = 0.5317
\]

\[
= 17.951
\]

The height of the tree is approximately 17.95 m.

**EXAMPLE:** Angle of Depression. An observer in a 55ft high lighthouse spots a ship at an angle of depression of 9.5° from their line of sight. How far is the ship from the lighthouse?
We are looking for the adjacent side of the angle so we will use the tangent ratio again.

\[
\tan 9.5^\circ = \frac{55}{x} \\
x = \frac{55}{\tan 9.5^\circ} \\
x \approx 328.7\text{ ft}
\]

**IMPORTANT TO NOTE:** The key to solving these problems is to DRAW A DIAGRAM before doing anything else! Once you draw your diagram, then decide which trig ratio will help you out and solve the problem!

**You Try!**

a) A boat is sitting 500m from the base of a cliff. Jade looks up at the top of the cliff at an angle of elevation of 32°. How high is the cliff?

b) A plane begins to descend at an angle of depression of 12°. The plane is a horizontal distance of 125km from the airport. How high is the plane when it begins its descent?

**Internet & Video Links**

Section 6.5 Homework Questions

Use the diagram to fill in the blanks below.

1. \( \tan D = \)  
2. \( \sin F = \)  
3. \( \tan F = \)  
4. \( \cos F = \)  
5. \( \sin D = \)  
6. \( \cos D = \)  

From questions 1-6, we can conclude the following. Fill in the blanks.

7. \( \cos \) \( = \) \( \sin F \) and \( \sin \) \( = \) \( \cos F \)  
8. The sine of an angle is \( \) to the cosine of its \( \)  
9. \( \tan D \) and \( \tan F \) are \( \) of each other.

Use your calculator to find the value of each trig function below. Round to four decimal places.

10. \( \sin 24^{\circ} \)  
11. \( \cos 45^{\circ} \)  
12. \( \tan 88^{\circ} \)  
13. \( \sin 43^{\circ} \)  

Find the sine, cosine and tangent of \( \angle A \). Reduce all fractions and radicals.

14.  

58
15.

Find the length of the missing sides. Round your answers to the nearest hundredth.

16.

17.

18.

19.

20.
21.

22.

23. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point, is 250 feet. How far away is she from the reef?

24. The Leaning Tower of Pisa currently “leans” at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?

25. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?
26. William spots a tree directly across the river from where he is standing. He then walks 20 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 65°. How wide is the river?

27. Diego is flying his kite one afternoon and notices that he has let out the entire 120 ft of string. The angle his string makes with the ground is 52°. How high is his kite at this time?

28. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?

29. Upon descent an airplane is 20,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.

30. **Critical Thinking** Why are the sine and cosine ratios always be less than 1?
Section 6.6: Inverse Trigonometric Ratios

INSTANT RECALL:

Find the lengths of the missing sides. Round your answer to the nearest hundredth.

1. 

2. 

3. Draw an isosceles right triangle with legs of length 3. What is the hypotenuse?

4. Use the triangle from #3, to find the sine, cosine, and tangent of $45^\circ$.

5. Explain why $\tan 45^\circ = 1$.

6.6a Inverse Trigonometric Ratios

The word *inverse* is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to “undo” it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities.

When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides, using sine, cosine or tangent. **With the inverse trig ratios, you can find the angle measure, given two sides.**

The inverse trig functions essentially are asking the question, “**what measure angle has the given side ratios?**” Remember, trig ratios are all
based on the similar sides so the actual side lengths don’t matter because they will give the same trig value. For example, if a right triangle has a sine ratio is \( \frac{2}{4} \) and another has a sine ratio of \( \frac{10}{20} \) both of these give a side ratio of 0.5. When we take the inverse sine of this value we find that the angle that both of these triangles equates to is 30°!

- **Inverse Tangent**: If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle.

Inverse tangent is also called arctangent and is labeled \( \tan^{-1} \) or \( \arctan \). The “-1” indicates inverse and NOT the negative reciprocal of the function.

- **Inverse Sine**: If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle.

Inverse sine is also called arcsine and is labeled \( \sin^{-1} \) or \( \arcsin \).

- **Inverse Cosine**: If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle.

Inverse cosine is also called arccosine and is labeled \( \cos^{-1} \) or \( \arccos \).

To find the inverse of a trig function you need to use your calculator. As we saw in the previous section on trigonometry and calculator use, to access the inverse function you will hit the trig button next to the equal sign and the same window will come up.

<table>
<thead>
<tr>
<th>( \sin )</th>
<th>( \cos )</th>
<th>( \tan )</th>
<th>( \csc )</th>
<th>( \sec )</th>
<th>( \cot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^{-1} )</td>
<td>( \cos^{-1} )</td>
<td>( \tan^{-1} )</td>
<td>( \csc^{-1} )</td>
<td>( \sec^{-1} )</td>
<td>( \cot^{-1} )</td>
</tr>
</tbody>
</table>

The bottom row are all of your inverse trig function. Select the inverse function you need to use and type in the value of the ratio. The screen shot below shows you the examples for inverse sine from above. We also see that...
no matter what the side lengths, as long as they simplify to 0.5 then the corresponding angles are always 30°.

We can therefore summarize arcsine as follows:

**If sin30° = 0.5 then the arcsin0.5 = 30°**

**YOU TRY!** Complete the following statements to show a trig function and its inverse

a) If sin45° = 1 then the arcsin 1 = ?

b) If arccos \( \frac{\sqrt{3}}{2} \) = 30° then the cos30° = ?

c) If tan60° = \( \sqrt{3} \) then the arctan \( \sqrt{3} \) = ?

d) If sin90° = 1 then the arcsin 1 = ?

e) If arccos 1 = 0° then the cos0° = ?

**Example:** In the diagram below we are given the opposite and adjacent sides of the reference angle A so we know we need to use the arctangent to find the value of angle A.
In our calculator we will input the following

So angle A is \(38.66^\circ\). Notice, in this example we converted the fraction to a decimal when inputting it into the calculator.

**Internet & Video Links**

2. [http://www.purplemath.com/modules/invratio.htm](http://www.purplemath.com/modules/invratio.htm)

**YOU TRY!**

Find the measure of the requested angle using the sides information given in the diagram.

a)
For each of the following, \( \angle A \) is an acute angle in a right triangle. Use your calculator to find \( m\angle A \) to the nearest tenth of a degree.

a) \( \text{arcsin } 0.68 = \) 

b) \( \text{arccos } 0.54 = \) 

c) \( \text{arctan } 0.34 = \)

The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft high. What is the angle of elevation, \( \theta \), of this escalator?

6.6b Solving Triangles

Now that we know how to use inverse trigonometric ratios to find the measure of the acute angles in a right triangle, we can solve right triangles. To solve a right triangle, you would need to find all sides and angles in a right triangle, using any method. When solving a right triangle, you could use sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem. Remember when solving right triangles you can only use the values that you are given.
Example: Solve the right triangle.

We already know that we can use the Pythagorean Theorem to find the side length of $c$ but to practice our trig, this example will approach it from a trig point of view. If you choose to use the Pythagorean Theorem to find the measure of side $c$, that is completely OK!

We know the $m\angle C$ will be the arccosine \( \frac{24}{30} \). Putting this in the calculator we get that $m\angle C$ is 36.87°. Through the triangle sum theorem we can than determine that $m\angle B = 180° - 90° - 36.87° = 53.13°$. Now to find and the measure of side $c$ we can use anything we want! Let’s use the sine ratio to help

\[
\sin 36.87° = \frac{c}{30} \\
c = 30 \cdot \sin 36.87° \\
c = 18.00
\]

We now know all three side lengths and all three angles that make up the right triangle ABC! Again, there are multiple ways to arrive at these answers don’t be afraid to find your own way!

YOU TRY!

a) Solve the right triangle.

b) Solve the right triangle.
6.6c Apply the Trig to Real Life!

Much like the trigonometric ratios, the inverse trig ratios can be used in several real-life situations. Here are a couple examples.

Example: The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115ft high. What is the angle of elevation, \( x^\circ \), of this escalator?

![Escalator Image]

Given the information in the problem, we have the opposite side and the hypotenuse so we must use the arcsine to find the measure of the angle.

Using a calculator we get \( \arcsin \frac{115}{230} = \arcsin 0.5 = 30^\circ \)!
YOU TRY!

a) A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?

b) Elise is standing on the top of a 50 foot building and spots her friend, Molly across the street. If Molly is 35 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise’s eye height is 4.5 feet.

Internet & Video Links

Section 6.6 Homework Questions

Use your calculator to find \( m\angle \)A to the nearest tenth of a degree.

1. 

2. 

3. 

4. 

5.
6. \( \triangle ABC \) is a right triangle. Find \( \angle A \) to the nearest tenth of a degree.

7. \( \sin A = 0.5684 \)
8. \( \cos A = 0.1234 \)
9. \( \tan A = 2.78 \)

Solving the following right triangles. Find all missing sides and angles.
14. Writing Explain when to use a trigonometric ratio to find a side length of a right triangle and when to use the Pythagorean Theorem.

15. Real-Life Situations Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

17. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?

18. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?

19. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?

20. Elizabeth wants to know the angle at which the sun hits a tree in her backyard at 3 pm. She finds that the length of the tree’s shadow is 24 ft at 3 pm. At the same time of day, her shadow is 6 ft 5 inches. If Elizabeth is 4 ft 8 inches tall, find the height of the tree and hence the angle at which the sunlight hits the tree.

21. Alayna is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 5 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
22. *Science Connection* Would the answer to number 20 be the same every day of the year? What factors would influence this answer? How about the answer to number 21? What factors might influence the path of the water?

23. Tommy was solving the triangle below and made a mistake. What did he do wrong?

\[
\text{tan}^{-1} \left( \frac{21}{28} \right) \approx 36.9^\circ
\]

24. Tommy then continued the problem and set up the equation:
\[
\cos 36.9^\circ = \frac{21}{x}
\]
By solving this equation he found that the hypotenuse was 26.3 units. Did he use the correct trigonometric ratio here? Is his answer correct? Why or why not?

25. How could Tommy have found the hypotenuse in the triangle another way and avoided making his mistake?

*Examining Patterns* Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Sine</th>
<th>Cosine</th>
<th>Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>0.1736</td>
<td>0.9848</td>
<td>0.1763</td>
</tr>
<tr>
<td>20°</td>
<td>0.3420</td>
<td>0.9397</td>
<td>0.3640</td>
</tr>
<tr>
<td>30°</td>
<td>0.5</td>
<td>0.8660</td>
<td>0.5774</td>
</tr>
<tr>
<td>40°</td>
<td>0.6428</td>
<td>0.7660</td>
<td>0.8391</td>
</tr>
<tr>
<td>50°</td>
<td>0.7660</td>
<td>0.6428</td>
<td>1.1918</td>
</tr>
<tr>
<td>60°</td>
<td>0.8660</td>
<td>0.3420</td>
<td>1.7321</td>
</tr>
<tr>
<td>70°</td>
<td>0.9397</td>
<td>0.1736</td>
<td>2.7475</td>
</tr>
<tr>
<td>80°</td>
<td>0.9848</td>
<td>0.0364</td>
<td>5.6713</td>
</tr>
</tbody>
</table>

26. What value is equal to \( \sin 40^\circ \)?
27. What value is equal to \( \cos 70^\circ \)?
28. Describe what happens to the sine values as the angle measures increase.
29. Describe what happens to the cosine values as the angle measures increase.
30. What two numbers are the sine and cosine values between?

31. Find $\tan 85^\circ$, $\tan 89^\circ$, and $\tan 89.5^\circ$ using your calculator. Now, describe what happens to the tangent values as the angle measures increase.

32. Explain why all of the sine and cosine values are less than one. (hint: think about the sides in the triangle and the relationships between their lengths)
Section 6.7: Laws of Sines and Cosines

In this chapter, we have only applied the trigonometric ratios to right triangles. However, you can extend what we know about these ratios and derive the Law of Sines and the Law of Cosines. Both of these laws can be used with any type of triangle to find any angle or side within it. That means we can find the sine, cosine and tangent of angle that are greater than 90°, such as the obtuse angle in an obtuse triangle.

6.7a - Law of Sines

- Law of Sine: If \( \triangle ABC \) has sides of length, \( a, b, \) and \( c \), then

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

Looking at a triangle, the lengths \( a, b, \) and \( c \) are opposite the angles \( A, B, \) and \( C \) of the same letter. In order to use the Law of Sines, you MUST have one side-angle relation completely defined in order to find the missing pieces.
**Example:** Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.

In this diagram we are given $m\angle C$ and the length of side $c$ so we know we can use the law of sine to help us find the length of side $b$.

\[
\frac{\sin 38^\circ}{12} = \frac{\sin 85^\circ}{b}
\]
\[
b \cdot \sin 38^\circ = 12 \cdot \sin 85^\circ
\]
\[
b = \frac{12 \cdot \sin 85^\circ}{\sin 38^\circ}
\]
\[
b = 19.42
\]

We now need to use the same law to find the side $a$. Remember, this is not a right triangle so you CANNOT use the Pythagorean Theorem to find the missing side! By the triangle sum theorem however we do know that angle $A$ is $180^\circ - 38^\circ - 85^\circ = 57^\circ$. Using this and the law of sine we get

\[
\frac{\sin 38^\circ}{12} = \frac{\sin 57^\circ}{a}
\]
\[
a \cdot \sin 38^\circ = 12 \cdot \sin 57^\circ
\]
\[
a = \frac{12 \cdot \sin 57^\circ}{\sin 38^\circ}
\]
\[
a = 16.35
\]

We now have all of the sides and angles defined for this non-right angle!

**YOU TRY!**

Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.
6.7b Law of Cosines

Law of Cosines: If \( \triangle ABC \) has sides of length \( a, b, \) and \( c \), then

\[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos A \\
b^2 &= a^2 + c^2 - 2ac \cos B \\
c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

Even though there are three formulas, they are all very similar. First, notice that whatever angle is in the cosine, the opposite side is on the other side of the equal sign. **We know we need to use the law of cosine because we are given two side lengths and only one angle with no opposing side!**

**Example:** Solve the triangle using Law of Cosines. Round your answers to the nearest hundredth.

![Triangle](image)

Since we are looking for side \( b \) we will use the second formula to solve this problem

\[
\begin{align*}
b^2 &= a^2 + c^2 - 2ac \cdot \cos B \\
b^2 &= 26^2 + 18^2 - 2(18)(26) \cdot \cos 26^\circ \\
b^2 &= 676 + 324 - 936 \cdot 0.8988 \\
b^2 &= 158.72 \\
\sqrt{b^2} &= \sqrt{158.72} \\
b &= 12.6
\end{align*}
\]

To find the remaining side you can then use either the Law of Cosine again or switch to the law of sines since you now have an angle side pair.

\[
\begin{align*}
\frac{\sin 26^\circ}{12.6} &= \frac{\sin A}{26} \\
\sin A &= (26 \cdot \sin 26^\circ)/12.6 \\
\sin A &= 0.9046 \\
\arcsin 0.9046 &= 64.77^\circ \\
m\angle A &= 64.77^\circ
\end{align*}
\]
YOU TRY!

Solve the triangle. Round your answers to the nearest hundredth.

To Summarize

Use Law of Sines when given:

- An angle and its opposite side.
- Any two angles and one side.
- Two sides and the non-included angle.

Use Law of Cosines when given:

- Two sides and the included angle.
- All three sides.
Section 6.7 Homework Questions

Use the Law of Sines or Cosines to solve \( \triangle ABC \). If you are not given a picture, draw one. Round all decimal answers to the nearest tenth.

1. 

\[
\begin{array}{c}
A & B & C \\
54^\circ & 42^\circ & 9 \\
\end{array}
\]

2. 

\[
\begin{array}{c}
A & B & C \\
87^\circ & 46^\circ & 12 \\
\end{array}
\]

3. 

\[
\begin{array}{c}
A & B & C \\
102^\circ & 16 & 25 \\
\end{array}
\]

4. 

\[
\begin{array}{c}
A & B & C \\
28^\circ & 18 & 15 \\
\end{array}
\]

5. 

\[
\begin{array}{c}
A & B & C \\
49 & 24 & 33 \\
\end{array}
\]
6. 

7. 

8. 

9. 

10. \( m\angle A = 74^\circ, m\angle B = 11^\circ, BC = 16 \)

11. \( mA = 64^\circ, AB = 29, AC = 34 \)

12. \( m\angle C = 133^\circ, m\angle B = 25^\circ, AB = 48 \)
Use the Law of Sines to solve $\triangle ABC$ below.

13. $m\angle A = 20^\circ$, $AB = 12$, $BC = 5$

Recall that when we learned how to prove that triangles were congruent we determined that SSA (two sides and an angle not included) did not determine a unique triangle. When we are using the Law of Sines to solve a triangle and we are given two sides and the angle not included, we may have two possible triangles. Problem 14 illustrates this.

14. Let’s say we have $\triangle ABC$ as we did in problem 13. In problem 13 you were given two sides and the not included angle. This time, you have two angles and the side between them (ASA). Solve the triangle given that $m\angle A = 20^\circ$, $m\angle C = 125^\circ$, $AC = 8.4$

15. Does the triangle that you found in problem 14 meet the requirements of the given information in problem 13? How are the two different $m\angle C$ related? Draw the two possible triangles overlapping to visualize this relationship.

*It is beyond the scope of this text to determine when there will be two possible triangles, but the concept of the possibility is something worth noting at this time.*

Additional Chapter Resources

1. [http://www.mathsisfun.com/geometry/index.html](http://www.mathsisfun.com/geometry/index.html)
2. [https://www.etap.org/demo/geo1/lesson.html](https://www.etap.org/demo/geo1/lesson.html)
Chapter Review Questions

Solve the following right triangles using the Pythagorean Theorem, the trigonometric ratios, and the inverse trigonometric ratios. When possible, simplify the radical. If not, round all decimal answers to the nearest tenth.

1. 

2. 

3. 

4. 

5.
Determine if the following lengths make an acute, right, or obtuse triangle. If they make a right triangle, determine if the lengths are a Pythagorean triple.

10. 11, 12, 13
11. 16, 30, 34
12. 20, 25, 42
13. $10\sqrt{6}, 30, 10\sqrt{15}$
14. 22, 25, 31
15. 47, 27, 35
Find the value of $x$.

16.

![Diagram](https://via.placeholder.com/150)

17.

![Diagram](https://via.placeholder.com/150)

18.

19. The angle of elevation from the base of a mountain to its peak is $76^\circ$. If its height is 2500 feet, what is the distance a person would climb to reach the top? Round your answer to the nearest tenth.

20. Taylor is taking an aerial tour of San Francisco in a helicopter. He spots AT&T Park (baseball stadium) at a horizontal distance of 850 feet and down (vertical) 475 feet. What is the angle of depression from the helicopter to the park? Round your answer to the nearest tenth.

Use the Law of Sines and Cosines to solve the following triangles. Round your answers to the nearest tenth.

21.

![Diagram](https://via.placeholder.com/150)
Texas Instruments Resources

*In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See [http://www.ck12.org/flexr/chapter/9693](http://www.ck12.org/flexr/chapter/9693).*